

ON TWO PROPERTIES OF OPERATOR ALGEBRAS:  
LOGMODULARITY OF SUBALGEBRAS, EMBEDDABILITY INTO  $\mathcal{R}^\omega$

A Dissertation

by

KATERYNA IUSHCHENKO

Submitted to the Office of Graduate Studies of  
Texas A&M University  
in partial fulfillment of the requirements for the degree of

DOCTOR OF PHILOSOPHY

December 2011

Major Subject: Mathematics

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December 2011

Major Subject: Mathematics

## ABSTRACT

On Two Properties of Operator Algebras:

Logmodularity of Subalgebras,

Embeddability into  $\mathcal{R}^\omega$ . (December 2011)

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This dissertation is devoted to several questions that arise in operator algebra theory. In the first part of the work we study the dilations of homomorphisms of subalgebras to the algebras that contain them. We consider the question whether a contractive homomorphism of a logmodular algebra into  $B(\mathcal{H})$  is completely contractive, where  $B(\mathcal{H})$  denotes the algebra of all bounded operators on a Hilbert space  $\mathcal{H}$ . We show that every logmodular subalgebra of  $M_n(\mathbb{C})$  is unitary equivalent to an algebra of block upper triangular matrices, which was conjectured by V. Paulsen and M. Raghupathi. In particular, this shows that every unital contractive representation of a logmodular subalgebra of  $M_n(\mathbb{C})$  is automatically completely contractive.

In the second part of the dissertation we investigate certain matrices composed of mixed, second-order moments of unitaries. The unitaries are taken from  $C^*$ -algebras with moments taken with respect to traces, or, alternatively, from matrix algebras with the usual trace. These sets are of interest in light of a theorem of E. Kirchberg about Connes' embedding problem and provide a new approach to it.

Finally, we give a modification of I. Klep and M. Schweighofer's algebraic reformulation of Connes' embedding problem by considering the  $*$ -algebra of the countably generated free group. This allows us to consider only quadratic polynomials

in unitary generators instead of arbitrary polynomials in self-adjoint generators.

## ACKNOWLEDGMENTS

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## CHAPTER I

### INTRODUCTION

This dissertation is devoted to study several questions in operator algebra theory.

We start with some basic definitions and facts. Let  $\mathcal{B}$  be a unital  $C^*$ -algebra and  $\mathcal{A} \subseteq \mathcal{B}$  be a closed unital subalgebra. Denote by  $\mathcal{A}^{-1}$  the set of invertible elements in  $\mathcal{A}$ . A non-commutative generalization of logmodularity of closed subalgebras of continuous functions on a compact Hausdorff space was introduced in [3], see also [4]. Namely,  $\mathcal{A}$  is *logmodular* in  $\mathcal{B}$  if the set  $\{a^*a : a \in \mathcal{A}^{-1}\}$  is dense in the set of positive invertible elements of  $\mathcal{B}$ .

In [23], V. Paulsen and M. Raghupathi studied the question: when does a contractive homomorphism of a logmodular algebra into  $B(\mathcal{H})$  extend to the  $C^*$ -algebra it generates? Generalizing C. Foias and I. Suciu, [10], they show that a contractive homomorphism  $\pi : \mathcal{A} \rightarrow B(\mathcal{H})$  is completely contractive if and only if its second amplification  $\pi^{(2)} = \pi \otimes 1_2 : \mathcal{A} \otimes M_2(\mathbb{C}) \rightarrow B(\mathcal{H}) \otimes M_2(\mathbb{C})$  is contractive. It is still unknown whether there are logmodular algebras with contractive but not completely contractive homomorphisms. In particular, it is interesting to decide this question for  $H^\infty(\mathbb{D})$ , considered as a logmodular subalgebra of  $L^\infty(\mathbb{T})$ , and the algebra of upper triangular matrices on infinite dimensional Hilbert space.

The decomposition of Cholesky shows that an algebra of block upper triangular matrices is logmodular in  $M_n(\mathbb{C})$ . It was proved by V. Paulsen and M. Raghupathi in [23] that if  $\mathcal{A} \subseteq M_n(\mathbb{C})$  is logmodular and contains the set of all diagonal matrices  $D_n$  then it is unitary equivalent to an algebra of block upper triangular matrices. It was conjectured that algebras of block upper triangular matrices are the only

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This dissertation follows the style of Algebra and Discrete Mathematics.

subalgebras on  $M_n(\mathbb{C})$  that have the logmodularity property. In Chapter II we prove this conjecture in the affirmative. It was proved in [22] that all contractive homomorphisms of an algebra of block upper triangular matrices are completely contractive. Thus there are no logmodular subalgebras in  $M_n(\mathbb{C})$  that can provide us with examples of contractive but not completely contractive homomorphisms.

In Chapter III and Chapter IV we develop an approach to Connes' Embedding Conjecture. Alain Connes's Embedding Conjecture (shortly CEC) concerns a fundamental approximation property of tracial states on von Neumann algebras. Recall that a von Neumann algebra is a weakly closed self-adjoint subalgebra of bounded operators acting on a Hilbert space. A finite von Neumann algebra is one which possess a finite tracial state, i.e. a linear functional, subject to some axioms which generalize the properties of the usual trace on a matrix algebra. In particular, a tracial state must take non-negative values on positive operators and zero values on commutators.

The most simple example of finite von Neumann algebra is the hyperfinite  $II_1$ -factor, denoted by  $R$ , which is an inductive limit of finite dimensional von Neumann algebras. It can also be described as the von Neumann algebra arising from the regular representation of an icc amenable group. CEC states that every finite von Neumann algebra with normal faithful tracial state can be embedded into the ultrapower  $R^\omega$  in a trace-preserving way. This is well known to be equivalent to the question whether all elements of a  $II_1$ -factor have matricial microstates in the sense of free entropy. In other words, the conjecture states that an arbitrary finite tracial state on a  $C^*$ -algebra can be point-wise approximated by normalized matrix traces.

This conjecture attracted a lot of attention recently in works of Kirchberg [17], Hadwin [13], Radulescu [27], [28], Collins - Dykema [9], Brown [7], Bercovici-Collins-Dykema-Li-Timotin [2] and many others.



There are several reformulations of CEC involving sums of Hermitian squares, obtained in [13], [18].

The idea behind these reformulations is to reduce the problem to an algebraic problem whether every trace-positive polynomial modulo linear span of commutators can be represented as a sum of Hermitian squares, i.e. the elements of the form  $f^*f$ .

Klep and Schweighofer use non-commutative polynomials  $f(a_1, \dots, a_n)$  in self-adjoint variables, and call them trace-positive if  $\text{Tr } f(A_1, \dots, A_n) \geq 0$  for all contractive self-adjoint matrices  $A_j$ . They prove that CEC is equivalent to the statement that every trace positive polynomial can be represented as some element from the quadratic module generated by  $1 - a_1^2, \dots, 1 - a_n^2$  plus a sum of commutators from the free algebra generated by  $a_1, \dots, a_n$ . Radulescu used some power series instead of polynomials. In Chapter III we proved that Connes' embedding conjecture is equivalent to the statement that for any self-adjoint  $f$  in the  $*$ -algebra  $\mathcal{F}$  of the free group on countably many generators  $u_1, u_2, \dots$ , which is of the form  $f(u_1, \dots, u_n) = \alpha e + \sum_{i \neq j} \alpha_{ij} u_i^* u_j$ , the condition that

$$\text{Tr}(f(V_1, \dots, V_n)) \geq 0 \tag{1.1}$$

for every  $m \geq 1$  and every  $n$ -tuple of unitary matrices  $V_1, \dots, V_n \in U(m)$ , implies that for every  $\varepsilon > 0$ ,  $\varepsilon e + f = g + c$  where  $c$  is a sum of commutators in  $\mathcal{F}$  and  $g$  is a sum of Hermitian squares.

This reformulation enhances the reformulation of Klep and Schweighofer in two aspects. Firstly, it uses the well-known cone of Hermitian squares rather than a specialized quadratic module, secondly, it involves only quadratic polynomials rather than polynomials of arbitrary length.

Another approach to CEC is to study second order mixed moments of unitaries in a  $II_1$ -factor. To be more precise, let  $\mathcal{G}_n$  be the set of all matrices  $(\tau(U_i^* U_j))_{1 \leq i, j \leq n}$

as  $U_1, \dots, U_n$  runs over  $n$ -tuples of unitaries in all  $C^*$ -algebras  $A$  possessing a faithful tracial state  $\tau$ . Consider a similarly defined set  $\mathcal{F}_n$  as the closure of the set of all matrices of the form  $(\text{tr}_k(U_i^* U_j))_{1 \leq i, j \leq n}$  for some  $k \in \mathbb{N}$ , where  $U_j \in M_k(\mathbb{C})$  are unitaries and  $\text{tr}_k$  is the normalized trace on  $M_k(\mathbb{C})$ . Kirchberg in [17] proved that CEC is equivalent to the statement that  $\mathcal{G}_n$  coincides for all  $n$  with the set  $\mathcal{F}_n$ . A matrix  $A \in M_n(\mathbb{C})$  is called a *correlation* matrix if  $A \geq 0$  and  $\text{diag}(A) = (1, \dots, 1)$ . Denote the set of all correlation  $n \times n$  matrices by  $\Theta_n$ . Then we have the following inclusions  $\mathcal{F}_n \subseteq \mathcal{G}_n \subseteq \Theta_n$ .

In IV we consider the extreme points of the sets  $\mathcal{F}_n$ ,  $\mathcal{G}_n$  and  $\Theta_n$ . Note that the extreme points of the set of correlation matrices were studied by many authors [8, 19, 20] and others. Quite surprisingly all three sets  $\mathcal{F}_n$ ,  $\mathcal{G}_n$  and  $\Theta_n$  restricted to  $M_n(\mathbb{R})$  coincide for every  $n \in \mathbb{N}$ . We show that for complex the case  $\Theta_n = \mathcal{F}_n$  iff  $n \leq 3$ .

## CHAPTER II

### LOGMODULAR SUBALGEBRAS OF FINITE DIMENSIONAL $C^*$ -ALGEBRAS

#### 1 Definitions

Let  $\mathcal{B}$  be a unital  $C^*$ -algebra and  $\mathcal{A} \subseteq \mathcal{B}$  be a closed unital subalgebra. Denote by  $\mathcal{A}^{-1}$  the set of invertible elements in  $\mathcal{A}$ . A non-commutative generalization of logmodularity of closed subalgebras of continuous functions on a compact Hausdorff space was introduced in [3], see also [4]. Namely,  $\mathcal{A}$  is *logmodular* in  $\mathcal{B}$  if the set  $\{a^*a : a \in \mathcal{A}^{-1}\}$  is dense in the set of positive invertible elements of  $\mathcal{B}$ .

In [23], V. Paulsen and M. Raghupathi study the question of when a contractive homomorphism of a logmodular algebra into  $B(\mathcal{H})$  is completely contractive, where  $B(\mathcal{H})$  denotes the algebra of all bounded operators on a Hilbert space  $\mathcal{H}$ . Generalizing C. Foias and I. Suci, [10], they showed that a contractive homomorphism  $\pi : \mathcal{A} \rightarrow B(\mathcal{H})$  extends to a positive map on the enveloping  $C^*$ -algebra if and only if its second amplification  $\pi^{(2)} = \pi \otimes 1_2 : \mathcal{A} \otimes M_2(\mathbb{C}) \rightarrow B(\mathcal{H}) \otimes M_2(\mathbb{C})$  is contractive. It is still unknown if there are logmodular algebras with contractive but not completely contractive homomorphisms. In particular, it would be interesting to decide this question for  $H^\infty(\mathbb{D})$  considered as logmodular subalgebra of  $L^\infty(\mathbb{T})$  and the algebra of upper triangular matrices on infinite dimensional Hilbert space.

The decomposition of Cholesky shows that an algebra of block upper triangular matrices is logmodular in  $M_n(\mathbb{C})$ . It was proved by V. Paulsen and M. Raghupathi in [23] that if  $\mathcal{A} \subseteq M_n(\mathbb{C})$  is logmodular and contains the diagonal matrices  $D_n$  then it is unitary equivalent to an algebra of block upper triangular matrices. It was conjectured that algebras of block upper triangular matrices are the only subalgebras on  $M_n(\mathbb{C})$  that have the logmodularity property. In Theorem 6 we prove this in the

affirmative. It was proved in [22] that all contractive homomorphisms of an algebra of block upper triangular matrices are completely contractive. Thus there are no logmodular subalgebras in  $M_n(\mathbb{C})$  that can provide us with examples of contractive but not completely contractive homomorphisms.

We will use the following notations. Matrix units of  $M_n(\mathbb{C})$  will be denoted by  $E_{i,j}$ . Put  $P_{\{i_1, \dots, i_n\}} = \sum_{i \in \{i_1, \dots, i_n\}} E_{i,i}$ . Given matrix  $a = [a_{ij}] \in M_n(\mathbb{C})$  we denote  $a(i, j) = a_{ij}$ .

## 2 A description of the logmodular subalgebras in $M_n(\mathbb{C})$

The proof of the main result will be divided into several lemmas. Note that if  $\mathcal{A}$  is logmodular in  $\mathcal{B}$  then  $\mathcal{A}^*$  is also logmodular in  $\mathcal{B}$ . Assume  $\mathcal{A} \subseteq M_n(\mathbb{C})$  is logmodular, then  $\mathcal{A}$  is unital and by a compactness argument for every positive  $b \in M_n(\mathbb{C})$  there exist  $a, c \in \mathcal{A}$  such that  $b = a^*a = cc^*$ . In particular we have the following lemma.

**Lemma 1.** If  $\mathcal{A}$  is logmodular in  $M_n(\mathbb{C})$  then there are  $\alpha_{ij}, \beta_{ij} \in \mathbb{C}$ ,  $i, j \in \{1, \dots, n\}$  such that each row of  $[\alpha_{ij}]$  and each column of  $[\beta_{ij}]$  is non-zero and  $\sum_k \alpha_{ik} E_{i,k} \in \mathcal{A}$ ,  $\sum_k \beta_{kj} E_{k,j} \in \mathcal{A}$  for every  $i, j \in \{1, \dots, n\}$

*Proof.* By logmodularity of  $\mathcal{A}$  we have  $E_{i,i} = a_i a_i^* = b_i^* b_i$  for some  $a_i, b_i \in \mathcal{A}$ . Then we can put  $\alpha_{ij} = a_i(i, j)$  and  $\beta_{ij} = b_i(i, j)$  which satisfies the statement.  $\square$

**Lemma 2.** Let  $v = (v_1, \dots, v_n) \in \mathbb{C}^n$ ,  $\|v\| = 1$  and

$$A = \text{span} \left( \sum_{j=1}^n v_j E_{1,j}, \dots, \sum_{j=1}^n v_j E_{n,j} \right) \subset M_n(\mathbb{C})$$

$$B = \text{span} \left( \sum_{i=1}^n v_i E_{i,1}, \dots, \sum_{i=1}^n v_i E_{i,n} \right) \subset M_n(\mathbb{C})$$

Then there exist unitaries  $U_1, U_2 \in M_n(\mathbb{C})$  such that  $E_{1,1} \in U_1 \mathcal{A} U_1^*$  and  $E_{1,1} \in U_2 \mathcal{B} U_2^*$ .

*Proof.* Let  $U_1 \in M_n(\mathbb{C})$  be unitary such that  $vU_1 = (1, 0, \dots, 0)$  then

$$\begin{aligned} U_1^*AU_1 &= \{U_1^* \begin{pmatrix} \alpha_1 v_1 & \dots & \alpha_1 v_n \\ & \dots & \\ \alpha_n v_1 & \dots & \alpha_n v_n \end{pmatrix} U_1 : \alpha_1, \dots, \alpha_n \in \mathbb{C}\} = \\ &= \{U_1^* \begin{pmatrix} \alpha_1 vU \\ & \dots & \\ \alpha_n vU \end{pmatrix} : \alpha_1, \dots, \alpha_n \in \mathbb{C}\} = \\ &= \{U_1^* \begin{pmatrix} \alpha_1 & 0 & \dots & 0 \\ & \dots & & \\ \alpha_n & 0 & \dots & 0 \end{pmatrix} : \alpha_1, \dots, \alpha_n \in \mathbb{C}\} \subseteq M_n(\mathbb{C})E_{1,1}. \end{aligned}$$

Since  $\dim(A) = n$  we have  $U_1^*AU_1 = M_n(\mathbb{C})E_{1,1}$  and therefore  $E_{1,1} \in U_1^*AU_1$ . The same argument applied to  $B^*$  provides the existence of  $U_2$ .  $\square$

In order to state the next lemma we need some definitions from general graph theory. A directed graph  $(V, \mathbb{R})$  is *transitive* if  $(i, j) \in \mathbb{R}$  and  $(j, k) \in \mathbb{R}$  implies  $(i, k) \in \mathbb{R}$  for every  $i, j, k \in V$ . A directed graph  $(V, \mathbb{R})$  is *full graph* if  $(i, j) \in \mathbb{R}$  for every  $i, j \in V$ . We will say that a subgraph  $(V_1, \mathbb{R}_1)$  of  $(V, \mathbb{R})$  *does not have an exit* if conditions  $i \in V_1, j \in V$  and  $(i, j) \in \mathbb{R}$  imply  $j \in V_1$ . A vertex  $i \in V$  has *an exit* if there is  $j \in V, j \neq i$  such that  $(i, j) \in \mathbb{R}$ .

**Lemma 3.** Let  $(V, \mathbb{R})$  be a finite directed graph. Assume that the graph  $(V, \mathbb{R})$  is transitive and every  $i \in V$  has an exit. Then  $(V, \mathbb{R})$  contains a full subgraph  $(V_1, \mathbb{R}_1)$  which does not have an exit.

*Proof.* Note that by the transitivity of  $\mathbb{R}$ , for every cycle  $\mathcal{C}$  in  $(V, \mathbb{R})$  the subgraph induced on  $\mathcal{C}$  by  $\mathbb{R}$  is automatically a full graph. Let  $i_0 \in V$  be an arbitrary vertex. Then there is an exit  $i_1 \in V$  with  $(i_0, i_1) \in \mathbb{R}$ . Let  $i_2 \in V$  be an exit of  $i_1$ . Continuing

this process we obtain a directed path with vertices  $i_0, \dots, i_n \in V$ . Since the graph is finite we will have a cycle in our sequence. Assume that this cycle is maximal in the sense that it is not contained in any other full subgraph of  $(V, \mathbb{R})$ . Then if this cycle has a vertex  $i$  with an exit, we will repeat the procedure with  $i_0 = i$ . Note that we can not return to any of previous sequences since that would contradict the maximality of the cycles that have been obtained before. Since the graph is finite we will eventually reach a cycle without any exit.  $\square$

**Lemma 4.** Let  $v_1, \dots, v_n$  be non-zero vectors in  $\mathbb{C}^n$  and  $\mathcal{A}$  be the algebra generated by

$$A_1 = \sum_{j=1}^n v_1(j) E_{1,j}, \dots, A_n = \sum_{j=1}^n v_n(j) E_{n,j}$$

Then there exist  $k \in \{1, \dots, n\}$ , a non-zero  $v \in \mathbb{C}^k$  and unitary  $U \in M_n(\mathbb{C})$  such that

$$\sum_{j=1}^k v(j) E_{1,j}, \dots, \sum_{j=1}^k v(j) E_{k,j} \in U \mathcal{A} U^*$$

*Proof.* Denote by  $\mathbb{R} \subseteq \{1, \dots, n\} \times \{1, \dots, n\}$  the set of indices with the property that  $(i, j) \in \mathbb{R}$  iff  $S(i, j) \neq 0$  for some  $S$  which is a word in the generators  $A_1, \dots, A_n$ . Thus if  $(i, j) \in \mathbb{R}$  and  $(j, k) \in \mathbb{R}$  then  $(i, k) \in \mathbb{R}$ . Since  $v_1, \dots, v_n$  are all non-zero we have that for every  $m \in \{1, \dots, n\}$  there exists  $i \in \{1, \dots, n\}$  such that  $(m, i) \in \mathbb{R}$ .

Consider the graph  $(\{1, \dots, n\}, \mathbb{R})$ . Let  $\mathbb{R}_1$  be the set of edges of a connected component of the directed graph  $(\{1, \dots, n\}, \mathbb{R})$  and let  $V$  be the set of vertices of  $\mathbb{R}_1$ . Then Lemma 3 implies that  $\mathbb{R}_1$  contains a full graph without exit, denote the set of its vertices by  $W = \{i_1, \dots, i_k\}$  and the set of its edges by  $\mathbb{R}'$ . Choose a unitary  $U$  that maps  $e_{i_1}, \dots, e_{i_k}$  to  $e_1, \dots, e_k$  by a permutation of the basis elements.

Since  $\mathbb{R}'$  does not have any exit, we have that for every  $(i, j) \in \mathbb{R}'$  there exists a word  $S$  in the generators  $A_1, \dots, A_n$  such that  $S(i, j) \neq 0$  and  $S \in P_{\{i_1, \dots, i_k\}} M_n(\mathbb{C}) P_{\{i_1, \dots, i_k\}}$ .

Since  $(W, \mathbb{R}')$  is full for every  $t \in \{i_1, \dots, i_k\}$  and  $i \in \{i_1, \dots, i_k\}$  we have  $(i, t) \in \mathbb{R}_1$ . Let  $S_i$  be the word on the generators  $A_1, \dots, A_n$  that provides  $(i, t) \in \mathbb{R}_1$ , then  $S_i = E_{i,i}S_i$  and  $S_iS_t = S_i(i, t)E_{i,t}S_t$ . Thus we have the statement of the lemma with  $v = (S_t(t, i_1), \dots, S_t(t, i_k))$ .  $\square$

**Lemma 5.** Fix  $k \in \{1, \dots, n\}$  and let  $A \subseteq P_{\{1, \dots, k\}}M_n(\mathbb{C})$  be a set such that for every  $m \in \{1, \dots, k\}$  and any set of indices  $\{i_1, \dots, i_m\} \subseteq \{1, \dots, k\}$  we have  $E_{m,m} \in A$  and  $P_{\{i_1, \dots, i_m\}}A(1_n - P_{\{i_1, \dots, i_m\}}) \neq \{0\}$ . Then for every  $t \in \{1, \dots, k\}$  the algebra generated by  $A$  contains an element  $S$  such that  $S(t, l) \neq 0$  for some  $l \in \{k+1, \dots, n\}$ .

*Proof.* Let  $I = \{i : E_{i,i}\mathcal{B}(1_n - P_{\{1, \dots, k\}}) = 0, 1 \leq i \leq k\}$  where  $\mathcal{B}$  is the algebra generated by  $A$ . To reach a contradiction assume that  $I \neq \emptyset$ . Permuting the part of the basis  $e_1, \dots, e_k$  we can assume  $I = \{1, \dots, d\}$ . Since  $P_{\{1, \dots, k\}}A(1 - P_{\{1, \dots, k\}}) \neq 0$  we have  $d < k$ . Take  $T \in P_{\{1, \dots, d\}}A(1_n - P_{\{1, \dots, d\}})$  and  $T(i, j) \neq 0$  for some  $1 \leq i \leq d$ ,  $d < j \leq k$ . There exists  $P \in E_{j,j}\mathcal{B}(1 - P_{\{1, \dots, k\}})$  with  $P(j, l) \neq 0$  for some  $k < l \leq n$ . Then the  $(i, l)$ -th entry of  $E_{i,i}TE_{j,j}P \in E_{i,i}\mathcal{B}(1 - P_{\{1, \dots, k\}})$  is non-zero which contradicts  $i \in I$ .  $\square$

Now we are ready to prove our main result.

**Theorem 6.** If  $\mathcal{A} \subseteq M_n(\mathbb{C})$  is logmodular then  $\mathcal{A}$  is unitary equivalent to an algebra of block upper triangular matrices.

*Proof.* We will proceed by induction on the dimension,  $n$ . For  $n = 1$  the statement is trivial. Assume that all logmodular subalgebras in  $M_k(\mathbb{C})$ ,  $k < n$ , are unitary equivalent to block upper triangular matrices. Let  $\mathcal{A}$  be logmodular in  $M_n(\mathbb{C})$ .

By Lemma 1 we have that there are non-zero  $v_1, \dots, v_n \in \mathbb{C}^n$  such that  $\sum_{j=1}^n v_1(j)E_{1,j}, \dots, \sum_{j=1}^n v_n(j)E_{n,j} \in \mathcal{A}$ . Then by Lemma 4 there exist  $k \in \{1, \dots, n\}$ ,  $v \in \mathbb{C}^k$  with  $\|v\| = 1$  and a unitary  $V \in M_n(\mathbb{C})$  such that

$\sum_{j=1}^k v(j)E_{1,j}, \dots, \sum_{j=1}^k v(j)E_{k,j} \in V\mathcal{A}V^*$ . Thus by Lemma 2 we have  $E_{1,1} \in V\mathcal{A}V^*$  for some unitary  $U \in M_n(\mathbb{C})$ .

We will prove by induction that  $E_{1,1}, \dots, E_{n,n} \in U\mathcal{A}U^*$  for some unitary  $U \in M_n(\mathbb{C})$ . Assume that  $E_{1,1}, \dots, E_{k,k} \subseteq V\mathcal{A}V^*$  for some  $V \in \mathcal{U}_n(\mathbb{C})$  and  $k < n$ . Denote  $V\mathcal{A}V^*$  again by  $\mathcal{A}$ .

Firstly, assume the existence of the set  $\{i_1, \dots, i_m\} \subseteq \{1, \dots, k\}$  such that  $P_{\{i_1, \dots, i_m\}}a(1_n - P_{\{i_1, \dots, i_m\}}) = 0$  for every  $a \in \mathcal{A}$ . Denote  $\mathcal{B} = (1_n - P_{\{i_1, \dots, i_m\}})\mathcal{A}(1_n - P_{\{i_1, \dots, i_m\}})$  and  $\mathcal{C} = (1_n - P_{\{i_1, \dots, i_m\}})M_n(\mathbb{C})(1_n - P_{\{i_1, \dots, i_m\}})$ . Then  $\mathcal{B}$  is logmodular in  $\mathcal{C}$ . Indeed, let  $a \in \mathcal{C}_+$  then there exists  $b \in \mathcal{A}$  such that  $a = b^*b$ , but  $P = (1_n - P_{\{i_1, \dots, i_m\}}) \in \mathcal{A}$ , therefore  $a = (bP)^*bP$  and  $bP \in \mathcal{B}$ . Since  $\mathcal{C}$  is unitary equivalent to  $M_t(\mathbb{C})$  for some  $t < n$  we have that  $\mathcal{B}$  is unitary equivalent to block upper triangular matrices and thus  $D_n \subseteq U\mathcal{A}U^*$  for some unitary  $U$ .

Thus we arrive at the case that for every  $m \in \{1, \dots, k\}$  and every subset  $\{i_1, \dots, i_m\} \subseteq \{1, \dots, k\}$  there exist an element  $a \in \mathcal{A}$  such that  $P_{\{i_1, \dots, i_m\}}a(1_n - P_{\{i_1, \dots, i_m\}}) \neq 0$ . We claim that for  $P = P_{\{k+1, \dots, n\}} = 1_n - P_{\{1, \dots, k\}} \in \mathcal{A}$  the set  $P\mathcal{A}P \subset \mathcal{A}$  contains

$$A_1 = \sum_{j=k+1}^n v_1(j)E_{k+1,j}, \dots, A_{n-k} = \sum_{j=k+1}^n v_{n-k}(j)E_{n,j}$$

for some non-zero  $v_i \in \mathbb{C}^{n-k}$ ,  $1 \leq i \leq n-k$ . Then from the claim, Lemma 2 and Lemma 4 it follows that there exists a unitary  $U \in M_{n-k}(\mathbb{C})$  such that

$$E_{k+1,k+1} \in (1_k \oplus U)P\mathcal{A}P(1_k \oplus U^*).$$

Therefore  $(1_k \oplus U)\mathcal{A}(1_k \oplus U^*)$  contains  $E_{1,1}, \dots, E_{k+1,k+1}$  and by induction we have the statement of the theorem.



To prove the claim consider a decomposition of  $E_{t,t}$  in  $\mathcal{A}$  for  $k+1 \leq t \leq n$ , that is  $R_t R_t^* = E_{t,t}$ , where  $R_t \in \mathcal{A}$  is a matrix with all rows equal to zero except for the  $t$ -th row. We will find  $v_i \in \mathbb{C}^{n-k}$  by "shifting" non-zero elements from the set  $R_t P_{\{1,\dots,k\}}$  to the set  $P\mathcal{A}P$ . Note that, now  $P_{\{1,\dots,k\}}\mathcal{A} \subseteq \mathcal{A}$  satisfies the assumptions of Lemma 5.

Assume that  $R_t(t, i) \neq 0$  for some  $1 \leq i \leq k$  and  $k+1 \leq t \leq n$ . By Lemma 5 there exists  $S \in \mathcal{A}$  with  $S(i, t) \neq 0$ . Denote  $R_t E_{i,i} S$  again by  $R_t$ . Doing this process with all  $R_t$ ,  $k+1 \leq t \leq n$  we obtain a set of non-zero rows with the property  $(1_n - P_{\{1,\dots,k\}})R_t(1_n - P_{\{1,\dots,k\}}) = R_t \neq 0$  for every  $k+1 \leq t \leq n$ . Then the vectors  $v_i$  with  $v_i(j) = R_{i+k}(i+k, j)$ ,  $1 \leq i \leq n-k$  have the required property.  $\square$

As a consequence of Theorem 6 and the fact that all contractive homomorphisms of an algebra of block upper triangular matrices are completely contractive, see [22], we get the following corollary.

**Corollary 7.** If  $\mathcal{A}$  is a logmodular subalgebra in  $M_n(\mathbb{C})$  then every contractive unital homomorphism  $\pi : \mathcal{A} \rightarrow B(H)$  is completely contractive.

## CHAPTER III

### MATRICES OF UNITARY MOMENTS

#### 1 Basic definitions and facts

One fundamental question about operator algebras is Connes' embedding problem, which in its original formulation asks whether every  $\text{II}_1$ -factor  $\mathcal{M}$  embeds in the ultrapower  $R^\omega$  of the hyperfinite  $\text{II}_1$ -factor  $R$ . This is well known to be equivalent to the question of whether all elements of  $\text{II}_1$ -factors possess matricial microstates, (which were introduced by Voiculescu [32] for free entropy), namely, whether such elements are approximable in  $*$ -moments by matrices. Connes' embedding problem is known to be equivalent to a number of different problems, in large part due to a remarkable paper [17] of Kirchberg (See also the survey [21], and the papers [24], [27], [28], [7], [29], [9], [18], [30], [15] for results with bearing on Connes' embedding problem.)

In Proposition 4.6 of [17], Kirchberg proved that, in order to show that a finite von Neumann algebra  $\mathcal{M}$  with faithful tracial state  $\tau$  embeds in  $R^\omega$ , it would be enough to show that for all  $n$ , all unitary elements  $U_1, \dots, U_n$  in  $\mathcal{M}$  and all  $\varepsilon > 0$ , there is  $k \in \mathbf{N}$  and there are  $k \times k$  unitary matrices  $V_1, \dots, V_n$  such that  $|\tau(U_i^* U_j) - \text{tr}_k(V_i^* V_j)| < \varepsilon$  for all  $i, j \in \{1, \dots, n\}$ , where  $\text{tr}_k$  is the normalized trace on  $M_k(\mathbf{C})$ . (He also required  $|\tau(U_i) - \text{tr}_k(V_i)| < \varepsilon$ , but this formally stronger condition is easily satisfied by taking the  $n + 1$  unitaries  $U_1, \dots, U_n, U_{n+1} = I$  in  $\mathcal{M}$  finding  $k \times k$  unitaries  $\tilde{V}_1, \dots, \tilde{V}_{n+1}$ , so that  $|\tau(U_i^* U_j) - \text{tr}_k(\tilde{V}_i^* \tilde{V}_j)| < \varepsilon$ , and letting  $V_i = \tilde{V}_{n+1}^* \tilde{V}_i$ .) It is, therefore, of interest to consider the set of possible second-order mixed moments of unitaries in such  $(\mathcal{M}, \tau)$  or, equivalently, of unitaries in  $C^*$ -algebras with respect to tracial states. (See also [27], where some similar sets were considered by F. Rădulescu.)

**Definition 8.** Let  $\mathcal{G}_n$  be the set of all  $n \times n$  matrices  $X$  of the form

$$X = (\tau(U_i^* U_j))_{1 \leq i, j \leq n} \quad (3.1)$$

as  $(U_1, \dots, U_n)$  runs over all  $n$ -tuples of unitaries in all  $C^*$ -algebras  $A$  possessing a faithful tracial state  $\tau$ .

**Remark 9.** The set-theoretic difficulties in the phrasing of Definition 8 can be evaded by insisting that  $A$  be represented on a given separable Hilbert space. Alternatively, let  $\mathfrak{A} = C\langle U_1, \dots, U_n \rangle$  denote the universal, unital, complex  $*$ -algebra generated by unitary elements  $U_1, \dots, U_n$ . A linear functional  $\phi$  on  $\mathfrak{A}$  is positive if  $\phi(a^* a) \geq 0$  for all  $a \in \mathfrak{A}$ . By the usual Gelfand–Naimark–Segal construction, any such positive functional  $\phi$  gives rise to a Hilbert space  $L^2(\mathfrak{A}, \phi)$  and a  $*$ -representation  $\pi_\phi : \mathfrak{A} \rightarrow B(L^2(\mathfrak{A}, \phi))$ . Thus, the set  $\mathcal{G}_n$  equals the set of all matrices  $X$  as in (3.1) as  $\tau$  runs over all positive, tracial, unital, linear functionals  $\tau$  on  $\mathfrak{A}$ .

**Definition 1.** Let  $\mathcal{F}_n$  be the closure of the set

$$\left\{ (\operatorname{tr}_k(V_i^* V_j))_{1 \leq i, j \leq n} \mid k \in \mathbf{N}, V_1, \dots, V_n \in \mathcal{U}_k \right\},$$

where  $\mathcal{U}_k$  is the group of  $k \times k$  unitary matrices.

A *correlation matrix* is a complex, positive semidefinite matrix having all diagonal entries equal to 1. Let  $\Theta_n$  be the set of all  $n \times n$  correlation matrices. Clearly, we have

$$\mathcal{F}_n \subseteq \mathcal{G}_n \subseteq \Theta_n.$$

Kirchberg's result is that Connes' embedding problem is equivalent to the problem of whether  $\mathcal{F}_n = \mathcal{G}_n$  holds for all  $n$ .

**Proposition 1.** *For each  $n$ ,*

- (i)  $\mathcal{F}_n$  and  $\mathcal{G}_n$  are invariant under conjugation with  $n \times n$  diagonal unitary matrices and permutation matrices,
- (ii)  $\mathcal{F}_n$  and  $\mathcal{G}_n$  are compact, convex subsets of  $\Theta_n$ ,
- (iii)  $\mathcal{F}_n$  and  $\mathcal{G}_n$  are closed under taking Schur products of matrices.

*Proof.* Part (i) is clear. Note that  $\Theta_n$  is a norm-bounded subset of  $M_n(\mathbf{C})$ . That  $\mathcal{F}_n$  is closed is evident. That  $\mathcal{G}_n$  is closed follows from the description in Remark 9 and the fact that a pointwise limit of positive traces on  $\mathfrak{A}$  is a positive trace. This proves compactness. Convexity of  $\mathcal{F}_n$  follows from by observing that if  $V$  is a  $k \times k$  unitary and  $V'$  is a  $k' \times k'$  unitary, then for arbitrary  $\ell, \ell' \in \mathbf{N}$ ,

$$\underbrace{V \oplus \cdots \oplus V}_{\ell \text{ times}} \oplus \underbrace{V' \oplus \cdots \oplus V'}_{\ell' \text{ times}}$$

can be realized as a block-diagonal  $(k\ell + k'\ell') \times (k\ell + k'\ell')$  matrix whose normalized trace is

$$\frac{k\ell}{k\ell + k'\ell'} \operatorname{tr}_k(V) + \frac{k'\ell'}{k\ell + k'\ell'} \operatorname{tr}_{k'}(V').$$

Convexity of  $\mathcal{G}_n$  follows because a convex combination of positive traces on  $\mathfrak{A}$  is a positive trace. This proves (ii).

Closedness of  $\mathcal{F}_n$  under taking Schur products follows by observing that if  $V$  and  $V'$  are unitaries as above, then  $V \otimes V'$  is a  $kk' \times kk'$  unitary whose normalized trace is  $\operatorname{tr}_k(V) \operatorname{tr}_{k'}(V')$ . For  $\mathcal{G}_n$ , we observe that if  $U$  and respectively,  $U'$ , are unitaries in  $C^*$ -algebras  $A$  and  $A'$  having tracial states  $\tau$  and  $\tau'$ , then the spatial tensor product  $C^*$ -algebra  $A \otimes A'$  has tracial state  $\tau \otimes \tau'$  that takes value  $\tau(U)\tau'(U')$  on the unitary  $U \otimes U'$ . This proves (iii).  $\square$

Since it is important to decide whether we have  $\mathcal{F}_n = \mathcal{G}_n$  for all  $n$ , it is interesting

to learn more about the sets  $\mathcal{F}_n$ . A first question is whether  $\mathcal{F}_n = \Theta_n$  holds. In Section 2, we show that this holds for  $n = 3$  but fails for  $n \geq 4$ . The proof relies on a characterization of extreme points of  $\Theta_n$ , and it uses also the set  $\mathcal{C}_n$  of matrices of moments of commuting unitaries. In Section 3 we prove  $M_n(\mathbf{R}) \cap \Theta_n \subseteq \mathcal{F}_n$ , and some further results concerning  $\mathcal{C}_n$ . In Section 4, we show that  $\mathcal{F}_n$  has nonempty interior, as a subset of  $\Theta_n$ .

## 2 Extreme points of $\Theta_n$ and some consequences

The set  $\Theta_n$  of  $n \times n$  correlation matrices is embedded in the affine space consisting of the self-adjoint complex matrices having all diagonal entries equal to 1; it is just the intersection of the set of positive, semidefinite matrices with this space. Every element of  $\Theta_n$  is bounded in norm by  $n$  (*cf* Remark 13), and  $\Theta_n$  is a compact, convex space. Since, in the space of self-adjoint matrices, every positive definite matrix is the center of a ball consisting of positive matrices, it is clear that the boundary of  $\Theta_n$  (for  $n \geq 2$ ) consists of singular matrices.

The extreme points of  $\Theta_n$  and  $\Theta_n \cap M_n(\mathbf{R})$  have been studied in [8], [20], [12] and [19]. In this section, we will use an easy characterization of the extreme points of  $\Theta_n$  to draw some conclusions about matrices of unitary moments. The papers cited above contain the facts about extreme points of  $\Theta_n$  found below, and have results going well beyond; the elementary techniques used here to characterize extreme points are essentially the same as used by Li and Tam [19]. In fact, we learned of these and the other results on correlation matrices only after our first version of this paper appeared. Because our presentation has a slightly different emphasis and these ideas are used later in examples, we provide the proofs, which are brief

We also introduce the subset  $\mathcal{C}_n$  of  $\mathcal{F}_n$ , consisting of matrices of moments of commuting unitaries. The new result in the section is Proposition 5, from which we

can conclude that there are no rank 2 extreme points of  $\mathcal{G}_n$  and, thus,  $\mathcal{G}_4 \neq \mathcal{F}_4$ .

This is a convenient place to recall the following standard fact. We include a proof for convenience.

**Lemma 10.** The set of all  $X \in \Theta_n$  of rank  $r$  is the set of all frame operators  $X = F^*F$  of frames  $F = (f_1, \dots, f_n)$ , consisting of  $n$  unit vectors  $f_j \in \mathbf{C}^r$ , where  $r = \text{rank}(X)$ . If, in addition,  $X \in M_n(\mathbf{R})$ , then the frame  $f_1, \dots, f_n$  can be chosen in  $\mathbf{R}^r$ .

*Proof.* Every frame operator  $F^*F$  as above clearly belongs to  $\Theta_n$  and has rank  $r$ .

Recall that the support projection of a Hermitian matrix  $X$  is the projection onto the orthocomplement of the nullspace of  $X$ . Suppose  $X \in \Theta_n$  has  $\text{rank}(X) = r$ . Let  $P$  be the support projection of  $X$  and let  $\lambda_1 \geq \dots \geq \lambda_r > 0$  be the nonzero eigenvalues of  $X$  with corresponding orthonormal eigenvectors  $g_1, \dots, g_r \in \mathbf{C}^n$ . Let  $V : \mathbf{C}^r \rightarrow P(\mathbf{C}^n)$  be the isometry defined by  $e_i \mapsto g_i$ , where  $e_1, \dots, e_r$  are the standard basis vectors of  $\mathbf{C}^r$ . So  $P = VV^*$ . Then  $X = F^*F$ , where  $F$  is the  $r \times n$  matrix

$$F = V^*X^{1/2} = \text{diag}(\lambda_1, \dots, \lambda_r)^{1/2}V^*.$$

If  $f_1, \dots, f_n \in \mathbf{C}^r$  are the columns of  $F$ , then  $\|f_i\| = X_{ii} = 1$  and the linear span of  $f_1, \dots, f_n$  is  $\mathbf{C}^r$ . Thus,  $f_1, \dots, f_n$  comprise a frame.

If  $X$  is real, then the vectors  $g_1, \dots, g_r$  can be chosen in  $\mathbf{R}^n$ . Then  $V$  and  $X^{1/2}$  are real matrices and  $f_1, \dots, f_n$  are in  $\mathbf{R}^r$ .  $\square$

**Lemma 11.** Let  $X \in M_n(\mathbf{C})$  be a positive semidefinite matrix and let  $P$  be the support projection of  $X$ . Then a Hermitian  $n \times n$  matrix  $Y$  has the property that there is  $\varepsilon > 0$  such that  $X + tY$  is positive semidefinite for all  $t \in (-\varepsilon, \varepsilon)$  if and only if  $Y = PYP$ .

*Proof.* If  $X = 0$  then this is trivially true, so suppose  $X \neq 0$ . After conjugating with a unitary, we may without loss of generality assume  $P = \text{diag}(1, \dots, 1, 0, \dots, 0)$  with

$\text{rank}(X) = \text{rank}(P) = r$ . Then  $PXP$ , thought of as an  $r \times r$  matrix, is positive definite. By continuity of the determinant, we see that if  $Y = PYP$ , then  $Y$  enjoys the property described above.

Conversely, if  $Y \neq PYP$ , then we may choose two standard basis vectors  $e_i$  and  $e_j$  for  $i \leq r < j$ , such that the compressions of  $X$  and  $Y$  to the subspace spanned by  $e_i$  and  $e_j$  are given by the matrices

$$\widehat{X} = \begin{pmatrix} x & 0 \\ 0 & 0 \end{pmatrix}, \quad \widehat{Y} = \begin{pmatrix} a & b \\ \bar{b} & c \end{pmatrix}$$

for some  $x > 0$ ,  $a, c \in \mathbf{R}$  and  $b \in \mathbf{C}$  with  $c$  and  $b$  not both zero. But

$$\det(\widehat{X} + t\widehat{Y}) = txc + t^2(ac - |b|^2).$$

If  $c \neq 0$ , then  $\det(\widehat{X} + t\widehat{Y}) < 0$  for all nonzero  $t$  sufficiently small in magnitude and of the appropriate sign, while if  $c = 0$  then  $b \neq 0$  and  $\det(\widehat{X} + t\widehat{Y}) < 0$  for all  $t \neq 0$ .  $\square$

**Proposition 2.** *Let  $n \in \mathbf{N}$ , let  $X \in \Theta_n$  and let  $P$  be the support projection of  $X$ . A necessary and sufficient condition for  $X$  to be an extreme point of  $\Theta_n$  is that there be no nonzero Hermitian  $n \times n$  matrix  $Y$  having zero diagonal and satisfying  $Y = PYP$ . Consequently, if  $X$  is an extreme point of  $\Theta_n$ , then  $\text{rank}(X) \leq \sqrt{n}$ .*

*Proof.*  $X$  is an extreme point of  $\Theta_n$  if and only if there is no nonzero Hermitian  $n \times n$  matrix  $Y$  such that  $X + tY \in \Theta_n$  for all  $t \in \mathbf{R}$  sufficiently small in magnitude. Now use Lemma 11 and the fact that  $\Theta_n$  consists of the positive semidefinite matrices with all diagonal values equal to 1.

For the final statement, if  $r = \text{rank}(X)$  then the set of Hermitian matrices with support projection under  $P$  is a real vector space of dimension  $r^2$ , while the space of  $n \times n$  Hermitian matrices with zero diagonal has dimension  $n^2 - n$ . If  $r^2 > n$ , then the intersection of these two spaces is nonzero.  $\square$

**Proposition 3.** *Let  $X \in \Theta_n$ . Suppose  $f_1, \dots, f_n$  is a frame consisting of  $n$  unit vectors in  $\mathbf{C}^r$ , where  $r = \text{rank}(X)$ , so that  $X = F^*F$  with  $F = (f_1, \dots, f_n)$  is the corresponding frame operator. (See Lemma 10.) Then  $X$  is an extreme point of  $\Theta_n$  if and only if the only  $r \times r$  self-adjoint matrix  $Z$  satisfying  $\langle Zf_j, f_j \rangle = 0$  for all  $j \in \{1, \dots, n\}$  is the zero matrix.*

*Proof.* Since  $F$  is an  $r \times n$  matrix of rank  $r$ , the map  $M_r(\mathbf{C})_{s.a.} \rightarrow M_n(\mathbf{C})_{s.a.}$  given by  $Z \mapsto F^*ZF$  is an injective linear map onto  $PM_n(\mathbf{C})_{s.a.}P$ , where  $P$  is the support projection of  $X$ . If  $Y = F^*ZF$ , then  $Y_{jj} = \langle Zf_j, f_j \rangle$ . Thus, the condition for  $X$  to be extreme now follows from the characterization found in Proposition 2.  $\square$

**Proposition 4.** *Let  $n \in \mathbf{N}$  and suppose  $X \in \Theta_n$  satisfies  $\text{rank}(X) = 1$ . Then  $X$  is an extreme point of  $\Theta_n$  and  $X \in \mathcal{F}_n$ . Moreover, using the notation introduced in Remark 9, we have*

$$\begin{aligned} \text{conv}\{X \in \Theta_n \mid \text{rank}(X) = 1\} = \\ = \{(\tau(U_i^*U_j))_{1 \leq i, j \leq n} \mid \tau : \mathfrak{A} \rightarrow \mathbf{C} \text{ a positive trace, } \tau(1) = 1, \pi_\tau(\mathfrak{A}) \text{ commutative}\} \end{aligned} \quad (3.2)$$

and this set is closed in  $\Theta_n$ .

**Notation 1.** We let  $\mathcal{C}_n$  denote the set given in (3.2). Thus, we have  $\mathcal{C}_n \subseteq \mathcal{F}_n$ . Moreover, (cf Remark 9),  $\mathcal{C}_n$  is the set of matrices as in (3.1) where  $(U_1, \dots, U_n)$  runs over all  $n$ -tuples of commuting unitaries in  $C^*$ -algebras  $A$  with faithful tracial state  $\tau$ .

*Proof of Proposition 4.* By Lemma 10, we have  $X = F^*F$  where  $F = (f_1, \dots, f_n)$  for complex numbers  $f_j$  with  $|f_j| = 1$ . Using Proposition 3, we see immediately that  $X$  is an extreme point of  $\Theta_n$ . Thinking of each  $f_j$  as a  $1 \times 1$  unitary, we have  $X \in \mathcal{F}_n$  and, moreover,  $X = (\tau(U_i^*U_j))_{1 \leq i, j \leq n}$ , where  $\tau : \mathfrak{A} \rightarrow \mathbf{C}$  is the character defined by



$\tau(U_i) = f_i$ ; in fact, it is apparent that every character on  $\mathfrak{A}$  yields a rank one element of  $\Theta_n$ . Since the set of traces  $\tau$  on  $\mathfrak{A}$  having  $\pi_\tau(\mathfrak{A})$  commutative is convex, this implies the inclusion  $\subseteq$  in (3.2).

That the left-hand-side of (3.2) is compact follows from Caratheodory's theorem, because the rank one projections form a compact set. If  $\tau : \mathfrak{A} \rightarrow \mathbf{C}$  is a positive trace with  $\tau(1) = 1$  and  $\pi_\tau(\mathfrak{A})$  commutative, then  $\tau = \psi \circ \pi_\tau$  for a state  $\psi$  on the  $C^*$ -algebra completion of  $\pi_\tau(\mathfrak{A})$ . Since every state on a unital, commutative  $C^*$ -algebra is in the closed convex hull of the characters of that  $C^*$ -algebra,  $\tau$  is itself the limit in norm of a sequence of finite convex combinations of characters of  $\mathfrak{A}$ .

Thus,  $X = (\tau(U_i^* U_j))_{1 \leq i, j \leq n}$  is the limit of a sequence of finite convex combinations of rank one elements of  $\Theta_n$ , and we have  $\supseteq$  in (3.2).  $\square$

**Remark 12.** We see immediately from (3.2) that  $\mathcal{C}_n$  is a closed convex set that is closed under conjugation with diagonal unitary matrices and permutation matrices; also, since the set of rank one elements of  $\Theta_n$  is closed under taking Schur products, so is the set  $\mathcal{C}_n$ . Furthermore, since  $\mathcal{C}_n$  lies in a vector space of real dimension  $m := n^2 - n$ , by Caratheodory's theorem every element of  $\mathcal{C}_n$  is a convex combination of not more than  $m + 1$  rank one elements of  $\Theta_n$ .

An immediate application of Propositions 2 and 4 is the following.

**Corollary 1.** *The extreme points of  $\Theta_3$  are precisely the rank one elements of  $\Theta_3$ . Moreover, we have*

$$\mathcal{C}_3 = \mathcal{F}_3 = \mathcal{G}_3 = \Theta_3.$$

**Remark 13.** Let  $X \in \mathcal{G}_n$  and take  $A$ ,  $\tau$  and  $U_1, \dots, U_n$  as in Definition 8 so that (3.1) holds, and assume without loss of generality that  $\tau$  is faithful on  $A$ . If we identify  $M_n(A)$  with  $A \otimes M_n(\mathbf{C})$ , then we have  $X = n(\tau \otimes \text{id}_{M_n(\mathbf{C})})(P)$ , where  $P$  is the

projection

$$P = \frac{1}{n} \begin{pmatrix} U_1^* \\ U_2^* \\ \vdots \\ U_n^* \end{pmatrix} (U_1 \ U_2 \ \cdots \ U_n)$$

in  $M_n(A)$ . If  $c = (c_1, \dots, c_n)^t \in \mathbf{C}^n$  is such that  $Xc = 0$ , then this yields  $\tau(Z^*Z) = 0$ , where  $Z = c_1U_1 + \cdots + c_nU_n$ . Since  $\tau$  is a faithful, we have  $Z = 0$ .

**Proposition 5.** *Let  $n \in \mathbf{N}$ . If  $X \in \mathcal{G}_n$  and  $\text{rank}(X) \leq 2$ , then  $X \in \mathcal{C}_n$ .*

*Proof.* If  $\text{rank}(X) = 1$ , then this follows from Proposition 4, so assume  $\text{rank}(X) = 2$ . Let  $\tau : \mathfrak{A} \rightarrow \mathbf{C}$  be a positive, unital trace such that  $X = (\tau(U_i^*U_j))_{1 \leq i, j \leq n}$  and let  $\pi_\tau : \mathfrak{A} \rightarrow B(L^2(\mathfrak{A}, \tau))$  be the  $*$ -representation as described in Remark 9. Let  $\sigma : \mathfrak{A} \rightarrow \pi_\tau(\mathfrak{A})$  be the  $*$ -representation defined by  $\sigma(U_i) = \pi_\tau(U_1)^* \pi_\tau(U_i)$  for each  $i \in \{1, \dots, n\}$  and let  $\tau' = \tau \circ \sigma$ . Then  $\tau'$  is a positive, unital trace on  $\mathfrak{A}$  and the matrix  $(\tau'(U_i^*U_j))_{1 \leq i, j \leq n}$  is equal to  $X$ . Furthermore,  $\pi_{\tau'}(U_1) = I$ . Consequently, we may without loss of generality assume  $\pi_\tau(U_1) = I$ .

Let  $e_1, \dots, e_n$  denote the standard basis vectors of  $\mathbf{C}^n$ . Let  $i, j \in \{2, \dots, n\}$ , with  $i \neq j$ . Since  $\text{rank}(X) = 2$ , there are  $c_1, c_i, c_j \in \mathbf{C}$  with  $c_1 \neq 0$  such that  $X(c_1e_1 + c_ie_i + c_je_j) = 0$ . By Remark 13, we have  $\pi_\tau(c_1I + c_iU_i + c_jU_j) = 0$ . We do not have  $c_i = c_j = 0$ , so assume  $c_i \neq 0$ . If  $c_j = 0$ , then  $\pi_\tau(U_i)$  is a scalar multiple of the identity, while if  $c_j \neq 0$ , then  $\pi_\tau(U_i)$  and  $\pi_\tau(U_j)$  generate the same  $\mathbf{C}^*$ -algebra, which is commutative. In either case, we have that the  $*$ -algebras generated by  $\pi_\tau(U_i)$  and  $\pi_\tau(U_j)$  commute with each other. Therefore,  $\pi_\tau(\mathfrak{A})$  is commutative, and  $X \in \mathcal{C}_n$ .  $\square$

**Corollary 2.**  $\mathcal{G}_4 \neq \Theta_4$ .

*Proof.* Combining Proposition 5 and Proposition 4, we see that  $\mathcal{G}_4$  has no extreme points of rank 2. It will suffice to find an extreme point  $X$  of  $\Theta_4$  with  $\text{rank}(X) = 2$ .

By Proposition 3, it will suffice to find four unit vectors  $f_1, \dots, f_4$  spanning  $\mathbf{C}^2$  such that the only self-adjoint  $Z \in M_2(\mathbf{C})$  satisfying  $\langle Zf_i, f_i \rangle = 0$  for all  $i = 1, \dots, 4$  is the zero matrix. It is easily verified that the frame

$$f_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad f_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad f_3 = \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}, \quad f_4 = \begin{pmatrix} i/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}$$

does the job, and, with  $F = (f_1, f_2, f_3, f_4)$ , this yields the matrix

$$X = F^*F = \begin{pmatrix} 1 & 0 & 1/\sqrt{2} & i/\sqrt{2} \\ 0 & 1 & 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} & 1 & (1+i)/2 \\ -i/\sqrt{2} & 1/\sqrt{2} & (1-i)/2 & 1 \end{pmatrix} \in \Theta_4 \setminus \mathcal{G}_4. \quad (3.3)$$

□

**Remark 14.** We cannot have  $\mathcal{C}_n = \mathcal{F}_n$  for all  $n$ , because by an easy a modification of Kirchberg's proof of Proposition 4.6 of [17], this would imply that  $M_2(\mathbf{C})$  can be faithfully represented in a commutative von Neumann algebra. (This argument shows that for some  $n$  there must be two-by-two unitaries  $V_1, \dots, V_n$  such that the matrix  $(\text{tr}_2(V_i^*V_j))_{1 \leq i, j \leq n}$  does not belong to  $\mathcal{C}_n$ .) In fact, in Proposition 8 we will show  $\mathcal{F}_6 \neq \mathcal{C}_6$ . However, we don't know whether  $\mathcal{F}_n = \mathcal{C}_n$  holds or not for  $n = 4$  or  $n = 5$ .

### 3 Real matrices

The main result of this section is the following, which easily follows from the usual representation of the Clifford algebra.

**Theorem 15.** For every  $n \in \mathbf{N}$ , we have

$$M_n(\mathbf{R}) \cap \Theta_n \subseteq \mathcal{F}_n.$$

We first recall the representation of the Clifford algebra. Let  $\Lambda$  be a linear map from a real Hilbert space  $H$  into the bounded, self-adjoint operators  $B(\mathcal{K})_{s.a.}$ , for some complex Hilbert space  $\mathcal{K}$ , satisfying

$$\Lambda(x)\Lambda(y) + \Lambda(y)\Lambda(x) = 2\langle x, y \rangle I_H, \quad (x, y \in H). \quad (3.4)$$

The real algebra generated by range of  $\Lambda$  is uniquely determined by  $H$  and called the real Clifford algebra.

Consider a real Hilbert space  $H$  of finite dimension  $r$  with its canonical basis  $\{e_i\}$ . Let

$$U = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad V = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Then the real Clifford algebra of  $H$  has the following representation by  $2^r \times 2^r$  matrixes

$$\Lambda(x) = \sum \lambda_i U^{\otimes i-1} \otimes V \otimes I_2^{\otimes (n-i)},$$

where  $x = \sum \lambda_i e_i$ . It easy to check that the relation (3.4) is satisfied. Moreover if  $\|x\| = 1$  then  $\Lambda(x)$  is symmetry, i.e.  $\Lambda(x)^* = \Lambda(x)$  and  $\Lambda(x)^2 = I$ .

*Proof of Theorem 15.* Let  $r$  be the rank of  $X$ . By Lemma 10, there are unit vectors  $f_1, \dots, f_n \in \mathbf{R}^r$  such that  $X_{i,j} = \langle f_i, f_j \rangle$  for all  $i$  and  $j$ . Taking  $\Lambda$  as described above, we get  $2^r \times 2^r$  unitary matrices  $\Lambda(f_i)$  (in fact, they are symmetries), and from (3.4) we have  $\text{tr}(\Lambda(f_i)\Lambda(f_j)) = \langle f_i, f_j \rangle$ .  $\square$

Below is the result for real matrices that is entirely analogous to Proposition 2.

**Proposition 6.** *Let  $n \in \mathbf{N}$ , let  $X \in M_n(\mathbf{R}) \cap \Theta_n$  and let  $P$  be the support projection of  $X$ . A necessary and sufficient condition for  $X$  to be an extreme point of  $M_n(\mathbf{R}) \cap \Theta_n$*

is that there be no nonzero Hermitian real  $n \times n$  matrix  $Y$  having zero diagonal and satisfying  $Y = PYP$ . Consequently, if  $X$  is an extreme point of  $M_n(\mathbf{R}) \cap \Theta_n$  and  $r = \text{rank}(X)$ , then  $r(r+1)/2 \leq n$ .

*Proof.* This is just like the proof of Proposition 2, the only difference being that the dimension of  $PM_n(\mathbf{R})_{s.a.}P$  for a projection  $P$  of rank  $r$  is  $r(r+1)/2$ .  $\square$

**Corollary 3.** *If  $n \leq 5$ , then*

$$M_n(\mathbf{R}) \cap \Theta_n \subseteq \mathcal{C}_n. \quad (3.5)$$

*Proof.* From Proposition 6, we see that every extreme point  $X$  of  $M_n(\mathbf{R}) \cap \Theta_n$  for  $n \leq 5$  has rank  $r \leq 2$ . But  $X \in \mathcal{F}_n \subseteq \mathcal{G}_n$ , by Theorem 15, so using Proposition 5, it follows that all extreme points of  $M_n(\mathbf{R}) \cap \Theta_n$  lie in  $\mathcal{C}_n$ . Since  $\mathcal{C}_n$  is closed and convex (see Proposition 4), the inclusion (3.5) follows.  $\square$

Of course, we also have the result for real matrices (and real frames) that is analogous to Proposition 3, which is stated below. The proof is the same.

**Proposition 7.** *Let  $X \in M_n(\mathbf{R}) \cap \Theta_n$ . Suppose  $f_1, \dots, f_n$  is a frame consisting of  $n$  unit vectors in  $\mathbf{R}^r$ , where  $r = \text{rank}(X)$ , so that  $X = F^*F$  with  $F = (f_1, \dots, f_n)$  is the corresponding frame operator. (See Lemma 10.) Then  $X$  is an extreme point of  $M_n(\mathbf{R}) \cap \Theta_n$  if and only if the only real Hermitian  $r \times r$  matrix  $Z$  satisfying  $\langle Zf_j, f_j \rangle = 0$  for all  $j \in \{1, \dots, n\}$  is the zero matrix.*

Although Corollary 3 shows that every element of  $M_n(\mathbf{R}) \cap \Theta_n$  for  $n \leq 5$  is in the closed convex hull of the rank one operators in  $\Theta_n$ , it is not true that every element of  $M_n(\mathbf{R}) \cap \Theta_n$  lies in the closed convex hull of rank one operators in  $M_n(\mathbf{R}) \cap \Theta_n$ , even for  $n = 3$ , as the following example shows.

**Example 16.** Consider the frame

$$f_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad f_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad f_3 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

of three unit vectors in  $\mathbf{R}^2$ . It is easily verified that the only real Hermitian  $2 \times 2$  matrix  $Z$  such that  $\langle Zf_i, f_i \rangle = 0$  for all  $i = 1, 2, 3$  is the zero matrix. Thus, by Proposition 7,

$$X = \begin{pmatrix} 1 & 0 & 1/\sqrt{2} \\ 0 & 1 & 1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} & 1 \end{pmatrix}$$

is a rank-two extreme point of  $M_3(\mathbf{R}) \cap \Theta_3$ . However, an explicit decomposition as a convex combination of rank one operators in  $\Theta_3$  is

$$X = \frac{1}{2} \begin{pmatrix} 1 & i & (1+i)/\sqrt{2} \\ -i & 1 & (1-i)/\sqrt{2} \\ (1-i)/\sqrt{2} & (1+i)/\sqrt{2} & 1 \end{pmatrix} + \frac{1}{2} \begin{pmatrix} 1 & -i & (1-i)/\sqrt{2} \\ i & 1 & (1+i)/\sqrt{2} \\ (1+i)/\sqrt{2} & (1-i)/\sqrt{2} & 1 \end{pmatrix}.$$

**Proposition 8.** *We have*

$$M_6(\mathbf{R}) \cap \Theta_6 \not\subseteq \mathcal{C}_6.$$

*Thus, we have  $\mathcal{F}_6 \neq \mathcal{C}_6$ .*

*Proof.* We construct an example of  $X \in (M_6(\mathbf{R}) \cap \Theta_6) \setminus \mathcal{C}_6$ . In fact, it will be a rank-three extreme point of  $M_6(\mathbf{R}) \cap \Theta_6$ .

Consider the frame

$$\begin{aligned} f_1 &= \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, & f_2 &= \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, & f_3 &= \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \\ f_4 &= \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, & f_5 &= \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, & f_6 &= \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \end{aligned}$$

of six unit vectors in  $\mathbf{R}^3$ . It is easily verified that the only real Hermitian  $3 \times 3$  matrix  $Z$  such that  $\langle Zf_i, f_i \rangle = 0$  for all  $i \in \{1, \dots, 6\}$  is the zero matrix. Thus, by Proposition 7,

$$X = \begin{pmatrix} 1 & 0 & 0 & 1/\sqrt{2} & 0 & 1/\sqrt{3} \\ 0 & 1 & 0 & 1/\sqrt{2} & 1/\sqrt{2} & 1/\sqrt{3} \\ 0 & 0 & 1 & 0 & 1/\sqrt{2} & 1/\sqrt{3} \\ 1/\sqrt{2} & 1/\sqrt{2} & 0 & 1 & 1/2 & \sqrt{2/3} \\ 0 & 1/\sqrt{2} & 1/\sqrt{2} & 1/2 & 1 & \sqrt{2/3} \\ 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} & \sqrt{2/3} & \sqrt{2/3} & 1 \end{pmatrix}$$

is a rank-three extreme point of  $M_6(\mathbf{R}) \cap \Theta_6$ . The nullspace of  $X$  is spanned by the vectors

$$\begin{aligned} v_1 &= (1/\sqrt{2}, 1/\sqrt{2}, 0, -1, 0, 0)^t \\ v_2 &= (0, 1/\sqrt{2}, 1/\sqrt{2}, 0, -1, 0)^t \\ v_3 &= (1/\sqrt{3}, 1/\sqrt{3}, 1/\sqrt{3}, 0, 0, -1)^t. \end{aligned}$$

Suppose, to obtain a contradiction, that we have  $X \in \mathcal{C}_6$ . Then there is a

commutative  $C^*$ -algebra  $A = C(\Omega)$  with a faithful tracial state  $\tau$  and there are unitaries  $I = U_1, U_2, \dots, U_6 \in A$  such that  $X = (\tau(U_i^* U_j))_{1 \leq i, j \leq 6}$ . Taking the vectors  $v_1, v_2$  and  $v_3$ , above, by Remark 13 we have

$$U_4 = \frac{1}{\sqrt{2}}(U_1 + U_2) \quad (3.6)$$

$$U_5 = \frac{1}{\sqrt{2}}(U_2 + U_3) \quad (3.7)$$

$$U_6 = \frac{1}{\sqrt{3}}(U_1 + U_2 + U_3). \quad (3.8)$$

Fixing any  $\omega \in \Omega$ , we have that  $\zeta_j := U_j(\omega)$  is a point on the unit circle  $\mathbf{T}$ , ( $1 \leq j \leq 6$ ). From (3.6) and  $|\zeta_4| = 1$ , we get  $\zeta_1 = \pm i\zeta_2$  and similarly from (3.7) we get  $\zeta_3 = \pm i\zeta_2$ . However, from (3.8), we then have

$$\zeta_6 \in \left\{ \frac{1-2i}{\sqrt{3}}\zeta_2, \frac{1}{\sqrt{3}}\zeta_2, \frac{1+2i}{\sqrt{3}}\zeta_2 \right\},$$

which contradicts  $|\zeta_6| = |\zeta_2| = 1$ . □

#### 4 Nonempty interior

In this section, we show that the interior of  $\mathcal{F}_n$  and, in fact, of  $\mathcal{C}_n$ , is nonempty, when considered as a subset of  $\Theta_n$ . (Since  $\mathcal{C}_n = \Theta_n$  for  $n = 1, 2, 3$ , this needs proving only for  $n \geq 4$ .)

Given  $X \in \Theta_n$ , let

$$a_X = \sup\{t \in [0, 1] \mid tX + (1-t)I \in \mathcal{F}_n\}$$

$$c_X = \sup\{t \in [0, 1] \mid tX + (1-t)I \in \mathcal{C}_n\}.$$

Of course,  $c_X \leq a_X$ . We now show that  $c_X$  is bounded below by a nonzero constant that depends only on  $n$ . In particular, we have that the identity element lies in the interior of  $\mathcal{C}_n$ , when this is taken as a subset of the affine space of self-adjoint matrices



having all diagonal entries equal to 1.

**Proposition 9.** *Let  $n \in \mathbb{N}$ ,  $n \geq 3$ , and let  $X \in \Theta_n$ . Then*

$$c_X \geq \frac{6}{n^2 - n}. \quad (3.9)$$

*Moreover, if  $\lambda_0$  is the smallest eigenvalue of  $X$ , then*

$$c_X \geq \min\left(\frac{6}{(n^2 - n)(1 - \lambda_0)}, 1\right). \quad (3.10)$$

*Proof.* We have  $X = (x_{ij})_{i,j=1}^n$  with  $x_{ii} = 1$  for all  $i = 1, \dots, n$ . Denote  $G = \{\sigma \in S_n \mid \sigma(1) < \sigma(2) < \sigma(3)\}$ . Then

$$\#G = \binom{n}{3}(n-3)!$$

Let  $U_\sigma = (u_{ij})$  be the permutation unitary matrix where  $u_{ij} = \delta_{i, \sigma(i)}$ . Then  $U^* X U = (x_{\sigma^{-1}(i) \sigma^{-1}(j)})_{i,j}$ . Define the block-diagonal matrix

$$B_\sigma = \begin{pmatrix} 1 & x_{\sigma(1)\sigma(2)} & x_{\sigma(1)\sigma(3)} \\ x_{\sigma(2)\sigma(1)} & 1 & x_{\sigma(2)\sigma(3)} \\ x_{\sigma(3)\sigma(1)} & x_{\sigma(3)\sigma(2)} & 1 \end{pmatrix} \oplus I_{n-3}.$$

Using Corollary 1 (and Remark 12), we easily see  $B_\sigma \in \mathcal{C}_n$ .

Let  $J_\sigma = \{(\sigma(1), \sigma(2)), (\sigma(1), \sigma(3)), (\sigma(2), \sigma(3))\}$ . Put  $X_\sigma = U^* B_\sigma U$ . Then

$$(X_\sigma)_{k\ell} = \begin{cases} 0, & \text{if } (k, \ell) \notin \{(1, 1), \dots, (n, n)\} \cup J_\sigma, \\ 1, & \text{if } k = \ell \\ x_{k\ell}, & \text{if } (k, \ell) \in J_\sigma. \end{cases}$$

Since for any  $k < \ell$  we have

$$\begin{aligned} \#\{\sigma \in G \mid \sigma(1) = k, \sigma(2) = \ell \text{ or } \sigma(1) = k, \sigma(3) = \ell \text{ or } \sigma(2) = k, \sigma(3) = \ell\} = \\ ((n - \ell) + (\ell - k - 1) + (k - 1))(n - 3)! = (n - 2)! \end{aligned}$$

it follows that matrix

$$X' = \frac{1}{\#G} \sum_{\sigma \in G} X_\sigma$$

has entries  $x'_{ii} = 1$ , and  $x'_{k\ell} = 6x_{k\ell}/(n^2 - n)$  if  $k \neq \ell$ .

Since  $\mathcal{C}_n$  is closed under conjugating with permutation matrices, we have  $X_\sigma \in \mathcal{C}_n$  for all  $\sigma \in G$ . But then the average  $X'$  also belongs to  $\mathcal{C}_n$ . This implies (3.9).

Now (3.10) is an easy consequence of (3.9). Indeed, if  $\lambda_0 = 1$ , then  $X$  is the identity matrix and  $c_X = 1$ . If  $\lambda_0 < 1$ , then let  $Y = (X - \lambda_0 I)/(1 - \lambda_0)$ . We have  $Y \in \Theta_n$ , and

$$(1 - t)I + tY = (1 - \frac{t}{1 - \lambda_0})I + \frac{t}{1 - \lambda_0}X.$$

This implies  $c_X \geq \min(1, c_Y/(1 - \lambda_0))$ . □

Given an  $n \times n$  matrix  $A = (a_{ij})_{1 \leq i, j \leq n}$ , let  $\bar{A}$  denote matrix whose  $(i, j)$  entry is the complex conjugate of  $a_{ij}$ . If  $A$  is self-adjoint, then so is  $\bar{A}$ , and these two matrices have the same eigenvalues (and multiplicities).

**Lemma 17.** Let  $X \in \Theta_n$  and let  $d > 0$  be such that

$$I + d \left( \frac{X - \bar{X}}{2} \right) \in \mathcal{F}_n.$$

Then  $a_X \geq d/(d + 1)$ . If  $n \leq 5$  and

$$I + d \left( \frac{X - \bar{X}}{2} \right) \in \mathcal{C}_n, \tag{3.11}$$

then  $c_X \geq d/(d + 1)$ .

*Proof.* The matrix  $(X + \overline{X})/2$  is real and lies in  $\Theta_n$ . Using Theorem 15, we have  $(X + \overline{X})/2 \in \mathcal{F}_n$ . Thus, we have

$$\frac{1}{d+1}I + \frac{d}{d+1}X = \frac{1}{d+1} \left( I + d \left( \frac{X - \overline{X}}{2} \right) \right) + \frac{d}{d+1} \left( \frac{X + \overline{X}}{2} \right) \in \mathcal{F}_n.$$

If  $n \leq 5$  and (3.11) holds, then we similarly apply Corollary 3.  $\square$

**Example 18.** Consider the matrix  $X$  as in (3.3), from Corollary 2. From Proposition 9 and closedness of  $\mathcal{F}_n$ , we know  $\frac{1}{2} \leq c_X \leq a_X < 1$ . It would be interesting to know the precise value of  $a_X$ , in order to have a concrete example of an element on the boundary of  $\mathcal{F}_4$  in  $\Theta_4$ .

Since

$$\frac{X - \overline{X}}{2} = \begin{pmatrix} 0 & 0 & 0 & i/\sqrt{2} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & i/2 \\ -i/\sqrt{2} & 0 & -i/2 & 0 \end{pmatrix}$$

has norm  $\sqrt{3}/2$  and since it is conjugate by a permutation matrix to an element of  $M_3(\mathbf{C}) \oplus \mathbf{C}$ , using Corollary 1 we have that (3.11) holds with  $d = 2/\sqrt{3}$ . A slightly better value is obtained by letting  $Y$  be the result of conjugation of  $X$  with the diagonal unitary  $\text{diag}(1, 1, 1, e^{-i\pi/4})$ . Then

$$\frac{Y - \overline{Y}}{2} = \begin{pmatrix} 0 & 0 & 0 & i/2 \\ 0 & 0 & 0 & -i/2 \\ 0 & 0 & 0 & 0 \\ -i/2 & i/2 & 0 & 0 \end{pmatrix}$$

which has norm  $1/\sqrt{2}$  and similarly yields  $d = \sqrt{2}$ . Applying Lemma 17 gives  $c_X = c_Y \geq \sqrt{2}/(1 + \sqrt{2}) \approx 0.586$ .

## CHAPTER IV

### ALGEBRAIC REFORMULATION OF CONNES' EMBEDDING PROBLEM AND THE FREE GROUP ALGEBRA

#### 1 Basic facts

Let  $\omega \in \beta(\mathbb{N}) \setminus \mathbb{N}$  be a free ultrafilter on  $\mathbb{N}$  and  $R$  be the hyperfinite  $\text{II}_1$ -factor with faithful tracial normal state  $\tau$ . Then the subset  $I_\omega$  in  $l^\infty(\mathbb{N}, R)$  consisting of  $(x_1, x_2, \dots)$  with  $\lim_{n \rightarrow \omega} \tau(x_n^* x_n) = 0$  is a closed ideal in  $l^\infty(\mathbb{N}, R)$  and a quotient algebra  $R^\omega = l^\infty(\mathbb{N}, R)/I_\omega$  is a von Neumann  $\text{II}_1$ -factor called *ultrapower* of  $R$ . It is naturally endowed with a faithful tracial normal state

$$\tau_\omega((x_n) + I_\omega) = \lim_{n \rightarrow \omega} \tau(x_n).$$

A. Connes' embedding problem asks whether every finite von Neumann algebra with fixed normal faithful tracial state can be embedded into  $R_\omega$  in a trace-preserving way.

It is well known that Connes' embedding problem is equivalent to the problem whether every finite set  $x_1, \dots, x_n$  of self-adjoint contractions in arbitrary  $\text{II}_1$ -factor  $(M, \tau)$  has *matricial microstates*, i.e. whether for any  $\varepsilon > 0$  and  $t \geq 1$  there is  $k \in \mathbb{N}$  and self-adjoint contractive  $k \times k$ -matrices  $A_1, \dots, A_n$  such that  $|\text{tr}(w(x_1, \dots, x_n)) - \tau(w(A_1, \dots, A_n))| < \varepsilon$  for all words  $w$  of length at most  $t$ .

In [13] D. Hadwin proved that solving Connes' embedding problem in affirmative is equivalent to proving that there is no polynomial  $p(x_1, \dots, x_n)$  in non-commutative variables such that

1.  $\text{tr}_k(p(A_1, \dots, A_n)) \geq 0$  for every  $k$  and self-adjoint contractions

$$A_1, \dots, A_n \in M_k.$$

2.  $\tau(p(T_1, \dots, T_n)) < 0$ , where  $T_1, \dots, T_n$  are self-adjoint contractive elements in a finite factor with trace  $\tau$ .

Recently I. Klep and M. Schweighofer established that Connes' embedding problem has the following equivalent algebraic reformulation.

Let  $f(X_1, \dots, X_m)$  be a self-adjoint element in a free associative algebra  $\mathbb{K}\langle \overline{X} \rangle$  with countable family of self-adjoint generators  $\overline{X} = \{X_1, X_2, \dots\}$ , where  $\mathbb{K} = \mathbb{R}$  or  $\mathbb{K} = \mathbb{C}$ . If  $\text{tr}(f(A_1, \dots, A_m)) \geq 0$  for any  $n$  and family of self-adjoint contractive matrices  $A_1, \dots, A_m \in M_n(\mathbb{K})$  then  $f$  has the property that for every  $\varepsilon > 0$  we have  $\varepsilon e + f = g + c$  where  $c$  is a sum of commutators in  $\mathbb{K}\langle \overline{X} \rangle$ ,  $g$  belongs to quadratic module generated by  $1 - X_i^2$  and  $e$  is the unit in  $\mathbb{K}\langle \overline{X} \rangle$ . Recall that a *quadratic module* is the smallest subset of  $\mathbb{K}\langle \overline{X} \rangle$  containing unit, closed under addition and conjugation  $x \rightarrow g^* x g$  by arbitrary  $g \in \mathbb{K}\langle \overline{X} \rangle$ .

In the present paper we consider the group  $*$ -algebra  $\mathcal{F}$  of the countably generated free group  $\mathbb{F}_\infty = \langle u_1, u_2, \dots \rangle$  instead of  $\mathbb{K}\langle \overline{X} \rangle$ . One reason is that we can use a more standard and well known set of hermitian squares  $\{g^* g | g \in \mathcal{F}\}$  instead of quadratic module  $M$  and the second that we can bound the degree of polynomials  $f$  in the above reformulation by 2. This modification provides the following.

**Theorem.** Connes' embedding conjecture is true iff for any self-adjoint  $f \in \mathcal{F}$  of the form  $f(u_1, \dots, u_n) = \alpha e + \sum_{i \neq j} \alpha_{ij} u_i^* u_j$  condition

$$\text{Tr}(f(V_1, \dots, V_n)) \geq 0 \quad (4.1)$$

for every  $m \geq 1$  and every  $n$ -tuple of unitary matrices  $V_1, \dots, V_n \in U(m)$  implies that for every  $\varepsilon > 0$ ,  $\varepsilon e + f = g + c$  where  $c$  is a sum of commutators and  $g$  is a sum of Hermitian squares.

We will call  $f$  satisfying (4.1) a *trace-positive quadratic polynomial*. Elements of

the form  $g + c$  with  $c$  being a sum of commutators are called *cyclically equivalent* to  $g$  (see Section 2).

## 2 An algebraic reformulation of the Connes' problem

Let  $\mathcal{F}$  be the  $*$ -algebra of the countably generated free group  $\mathbb{F}_\infty$ . Let  $K$  denote the  $\mathbb{R}$ -subspace in  $\mathcal{F}_{sa}$  generated by the commutators  $fg - gf$  ( $f, g \in F$ ). We will say that  $f$  and  $g$  in  $\mathcal{F}$  are *cyclically equivalent* (denote  $f \stackrel{cyc}{\sim} g$ ) if  $f - g \in K$ . Let  $\Sigma^2(\mathcal{F})$  denote the set of positive elements of the  $*$ -algebra  $\mathcal{F}$ , i.e. elements of the form  $\sum_{j=1}^m f_j^* f_j$  with  $f_j \in \mathcal{F}$ . An element of the form  $f^* f$  is called Hermitian square and therefore the cone  $\Sigma^2(\mathcal{F})$  is called the cone of Hermitian squares.

**Definition 19.** Let  $C$  be a subset of the vector space  $V$ . An element  $v \in C$  is called an algebraic interior point of  $C$  if for every  $u \in V$  there is  $\varepsilon > 0$  in  $\mathbb{R}$  s.t.  $v + \lambda u \in C$  for all  $0 \leq \lambda \leq \varepsilon$ .

**Definition 20.** Let  $\mathcal{A}$  be a unital  $*$ -algebra with the unit  $e$ . Then

1. An element  $a \in \mathcal{A}_{sa}$  is called bounded if there is  $\alpha \in \mathbb{R}_+$  such that  $\alpha e \pm a \in \Sigma^2(\mathcal{A})$ .
2. An element  $x = a + ib$  with  $a, b \in \mathcal{A}_{sa}$  is bounded if the elements  $a$  and  $b$  are such.
3. The algebra  $\mathcal{A}$  is bounded if all its elements are bounded.

It is well known that the set of all bounded elements in  $\mathcal{A}$  is a  $*$ -subalgebra in  $\mathcal{A}$  and that an element  $x \in \mathcal{A}$  is bounded if and only if  $xx^*$  is such (see for example [26, 16]). In particular  $\mathcal{F}$  is a bounded  $*$ -algebra. Obviously this implies that the unit of the algebra is an algebraic interior point of  $\Sigma^2(\mathcal{F})$ .

The following lemma is a modification of Theorem 3.12 in [18].

**Lemma 21.** Let  $f \in \mathcal{F}$  be self-adjoint. If for any  $\text{II}_1$  factor  $M$  with faithful normal tracial state  $\tau$  and separable predual and every  $n$ -tuple of unitary elements  $U_1, \dots, U_n$

in the unitary group  $\mathcal{U}(M)$  of  $M$  we have that

$$\tau(f(U_1, \dots, U_n)) \geq 0$$

then for every  $\varepsilon > 0$ ,  $\varepsilon e + f \sim g$  for some  $g \in \Sigma^2(\mathcal{F})$ .

*Proof.* Clearly  $\Sigma^2(\mathcal{F}) + K$  is a convex cone in  $\mathbb{R}$ -space  $\mathcal{F}_{sa}$ . Since  $e$  is an algebraic internal point of  $\Sigma^2(\mathcal{F})$  it is also an algebraic internal point of  $\Sigma^2(\mathcal{F}) + K$ .

Assume that there is  $\varepsilon > 0$  such that  $\varepsilon e + f \not\sim g$  for any  $g \in \Sigma^2(\mathcal{F})$ , i.e.  $\varepsilon e + f \notin \Sigma^2(\mathcal{F}) + K$ . By Eidelheit-Kakutani separation theorem there is  $\mathbb{R}$ -linear unital functional  $L_0 : \mathcal{F}_{sa} \rightarrow \mathbb{R}$  s.t.  $L_0(\Sigma^2(\mathcal{F}) + K) \subseteq \mathbb{R}_{\geq 0}$  and  $L_0(\varepsilon e + f) \in \mathbb{R}_{\leq 0}$ . Since  $-K \subset \Sigma^2(\mathcal{F}) + K$  we have that  $L_0(K) = 0$ . In particular extending  $L_0$  to  $\mathbb{C}$ -linear functional on  $\mathcal{F}$  we get a tracial functional  $L$ . Since  $L$  maps  $\Sigma^2(\mathcal{F})$  into the non-negative reals it defines a pre-Hilbert space structure on  $\mathcal{F}$  by means of sesquilinear for  $\langle p, q \rangle = L(q^*p)$ ,  $p, q \in \mathcal{F}$ . Let  $N = \{p : \langle p, p \rangle = 0\}$ . By Cauchy-Schwarz inequality  $N = \{p : L(q^*p) = 0 \text{ for all } q \in \mathcal{F}\}$  and hence is a left ideal. Let  $H_0$  be the pre-Hilbert space  $\mathcal{F}/N$ . Consider the left regular representation  $\pi : \mathcal{F} \rightarrow L(H_0)$ . Since  $\pi$  is a  $*$ -homomorphism for every  $f \in \mathcal{F}$  operator  $\pi(f)$  is bounded as a linear combination of unitary operators. Thus  $\pi(f)$  can be extended to the bounded operator acting on the Hilbert space  $H$  which is the completion of  $H_0$ . Thus we have a representation  $\pi : \mathcal{F} \rightarrow B(H)$  with a cyclic vector  $\xi = e + N$  and such that  $L(p) = \langle \pi(p)\xi, \xi \rangle$ . In particular  $L$  is a contractive tracial state on  $\mathcal{F}$  and thus defines a tracial state of the universal enveloping  $C^*$ -algebra  $C^*(\mathcal{F})$ . By Banach-Alaoglu and Krein-Milman theorem we can assume that  $L$  is an extreme point in the set of all tracial states and thus  $\pi(\mathcal{F})$  generates a factor von Neumann algebra  $M$  (see [13]). Clearly  $M$  is a finite factor. If it is type  $I$  then it should be  $\mathbb{C}$  (since  $\xi$  is a trace vector) and thus can be embedded into any  $\text{II}_1$ -factor in trace preserving way. Thus we can assume that  $M$

is a type  $\text{II}_1$ -factor. But then condition  $L(f) < 0$  is impossible.  $\square$

**Corollary 22.** If self-adjoint  $f \in \mathcal{F}$  has real coefficients and for any real type  $\text{II}_1$  von Neumann algebra  $(M, \tau)$  with normal faithful tracial state  $\tau$  and every  $n$ -tuple of unitary elements  $U_1, \dots, U_n$  in  $M$  we have that

$$\tau(f(U_1, \dots, U_n)) \geq 0$$

then the same holds for the complex  $\text{II}_1$  von Neumann algebras.

*Proof.* Element  $f$  can be written as  $f = \alpha + \sum_{w_j} \alpha_{w_j}(w_j + w_j^*)$  with  $\alpha_{w_j} \in \mathbb{R}$  and for complex trace  $\tau$  and  $U_1, \dots, U_n \in U(M)$  we will have  $\tau(f) = \alpha + 2 \sum_{w_j} \alpha_{w_j} \operatorname{Re} \tau(w_j)$ , i.e.  $\tau(f) = (\operatorname{Re} \tau)(f)$ . To finish the proof note that  $M$  can be regarded as a real finite von Neumann algebra with faithful trace  $\operatorname{Re} \tau$ .  $\square$

**Lemma 23.** If  $f \in \mathbb{R}[\mathbb{F}_\infty]$ ,  $f = f^*$  and for every real type  $\text{II}_1$  von Neumann algebra  $(M, \tau)$  we have that  $\tau(f) \geq 0$  then for every  $\varepsilon > 0$ ,  $\varepsilon + f \stackrel{cyc}{\sim} g$  for some  $g \in \left\{ \sum_{j=1}^m g_j^* g_j \mid m \in \mathbb{N}, g_j \in \mathbb{R}\langle \mathbb{F}_\infty \rangle \right\}$ .

*Proof.* The proof of this statement can be obtained by obvious modification of the proof of lemma 21. The only nontrivial part is that the unit  $e$  is an algebraic internal point but this is equivalent to  $\mathbb{R}\langle \mathbb{F}_\infty \rangle$  being bounded  $*$ -algebra. The proof of the last fact can be found in [31].  $\square$

This lemma gives another proof of corollary 22. In sequel we will need the following lemma.

**Lemma 24.** If  $(M, \tau)$  is a  $\text{II}_1$  factor which can be embedded into  $R^\omega$  and  $f \in \mathcal{F}$  is self-adjoint then the condition  $\operatorname{tr}(f(V_1, \dots, V_n)) \geq 0$  for all  $m \geq 0$  and all unitary  $V_1, \dots, V_n$  in  $M_{m \times m}(\mathbb{C})$  implies that  $\tau(f(U_1, \dots, U_n)) \geq 0$  for all unitary  $U_1, \dots, U_n$  in  $M$ .



*Proof.* Considering  $M$  as a subalgebra in  $R^\omega$  and  $\tau$  as a restriction of the trace on  $R^\omega$  we can find a representing sequences  $\{u_j^{(k)}\}_{j=1}^\infty$  for  $U_k$ ,  $k = 1, \dots, n$  in  $l^\infty(\mathbb{N}, R)$  which are unitary elements in von Neumann algebra  $l^\infty(\mathbb{N}, R)$ . This can be done since every unitary in von Neumann algebra  $R^\omega$  can be lifted to a unitary in von Neumann algebra  $l^\infty(\mathbb{N}, R)$  with respect to canonical morphism  $\pi : l^\infty(\mathbb{N}, R) \rightarrow R^\omega$ . Taking  $j$  sufficiently large we can approximate mixed moments of  $U_1, \dots, U_k$  up to order  $m$ , i.e.  $\tau(U_{s_1} \dots U_{s_t})$  with  $t \leq m$  and  $s_1, \dots, s_t \in \{1, \dots, n\}$ , by the mixed moments of unitary matrices  $u_1^{(k)}, \dots, u_n^{(k)}$ .  $\square$

The following theorem is Proposition 4.6 in [17]

**Theorem 25. (E. Kirchberg)** Let  $(M, \tau)$  be von Neumann algebra with separable predual and faithful normal tracial state  $\tau$ . If for all  $n \geq 1$  and for all unitaries  $u_1, \dots, u_n$  in  $M$  and for arbitrary  $\varepsilon > 0$  there exists  $m \geq 1$  and unitary  $m \times m$  matrices  $V_1, \dots, V_n \in U(m)$  s.t. for all  $i, j$ :

$$|\tau(u_i^* u_j) - \frac{1}{m} \text{Tr}(V_i^* V_j)| < \varepsilon, \quad (4.2)$$

$$|\tau(u_j) - \frac{1}{m} \text{Tr}(V_j)| < \varepsilon \quad (4.3)$$

then  $M$  can be embedded into  $R^\omega$ .

**Remark 26.** We may drop condition (4.3) since we may take  $u_0 = 1, u_1, \dots, u_n$  and by (4.2) find matrices  $W_0, \dots, W_n$  such that  $|\tau(u_i^* u_j) - (1/m) \text{Tr}(W_i^* W_j)| < \varepsilon$  for all  $i$  and  $j$ . Thus (4.2) and (4.3) will be satisfied if we take  $V_j = W_0^* W_j$ .

The proof of the following theorem is an adaptation of the proof of Proposition 3.17 from [18].

**Theorem 27.** Let  $(M, \tau)$  be  $II_1$ -factor with separable predual. If for every self-

adjoint element  $f \in \mathcal{F}$  of the form  $f = \alpha + \sum_{i \neq j} \alpha_{ij} u_i^* u_j$  the condition

$$\text{Tr}(f(V_1, \dots, V_n)) \geq 0$$

for all  $m \geq 1$  and every  $n$ -tuple of unitary matrices  $V_1, \dots, V_n \in U(m)$  implies that  $\tau(f(U_1, \dots, U_n)) \geq 0$  for all unitaries  $U_1, \dots, U_n$  in  $M$  then  $M$  can be embedded into  $R^\omega$ .

*Proof.* Take  $n \geq 1$ . Consider the finite dimensional vector space  $W = \{\alpha e + \sum_{i \neq j} \alpha_{ij} u_i^* u_j \mid \alpha_{ij} \in \mathbb{C}\}$ . Denote by  $C$  the convex hull of the set  $F$  of the functionals  $T \in W^*$  of the form  $T(p) = (1/m) \text{Tr}(p(V_1, \dots, V_n))$  where  $m \geq 1$  and  $V_1, \dots, V_n \in U(m)$ . Take arbitrary  $n$ -tuple of unitary elements  $U_1, \dots, U_n$  in  $M$  and put  $L(p) = \tau(p(U_1, \dots, U_n))$  for  $p \in W$ . Assume that  $L \notin C$ . By Hahn-Banach theorem there is  $f \in W^{**} = W$  and  $c \in \mathbb{R}$  s.t.  $\text{Re}(L(f)) < c < \text{Re}(T(f))$  for all  $T \in C$ . Since  $e \in W$  we can substitute  $f - c$  instead of  $f$  and thus assume that  $c = 0$ . Since  $T(f^*) = \overline{T(f)}$  for every  $T \in C$  and  $L(f^*) = \overline{L(f)}$  we have that  $L(f + f^*) = 2\text{Re}(L(f)) < 0 < 2\text{Re}(T(f)) = T(f + f^*)$  which is a contradiction. Thus  $L \in C$ . Let  $T$  be a rational convex combination of elements  $T_1, \dots, T_s$  from  $F$  and  $T_k$  corresponds to  $n$ -tuples  $V_{j,k}$ . Then  $T = (1/q)(p_1 T_1 + \dots + p_s T_s)$  for some positive integers  $p_1, \dots, p_s, q$ . Taking block-diagonal  $V_j = (V_{j,1}^{\otimes p_1} \oplus \dots \oplus V_{j,s}^{\otimes p_s})$  we see that  $T \in F$ . Thus each element of  $C$ , in particular element  $L$  can be approximated by elements of  $F$ . By the Kirchberg's Theorem we have that  $M$  can be embedded into  $R^\omega$ .

□

**Theorem 28.** Connes' embedding conjecture problem has affirmative solution iff for any self-adjoint  $f \in \mathcal{F}$  of the form  $f = \alpha e + \sum_{i \neq j} \alpha_{ij} u_i^* u_j$  condition

$$\text{Tr}(f(V_1, \dots, V_n)) \geq 0$$

for every  $m \geq 1$  and every  $n$ -tuple of unitary matrices  $V_1, \dots, V_n \in U(m)$  implies that for every  $\varepsilon > 0$ ,  $\varepsilon e + f \sim g$  with  $g \in \Sigma^2(\mathcal{F})$ .

*Proof.* If Connes' embedding problem has affirmative solution and quadratic  $f \in \mathcal{F}_{sa}$  is such that  $\text{Tr}(f(V_1, \dots, V_n)) \geq 0$  for every  $m \geq 1$  and every  $n$ -tuple of unitary matrices  $V_1, \dots, V_n \in U(m)$  then by lemma 24 we have  $\tau(f(U_1, \dots, U_n)) \geq 0$  for any unitary  $U_1, \dots, U_n$  in  $M$ . Hence by lemma 21,  $\varepsilon e + f$  is cyclically equivalent to a sum of Hermitian squares. This proves that the conditions of the theorem are necessary.

If  $\varepsilon e + f$  is cyclically equivalent to an element in  $\Sigma^2(\mathcal{F})$  for every  $\varepsilon > 0$  then clearly  $\tau(f(U_1, \dots, U_n)) \geq 0$  for any unitary  $U_1, \dots, U_n$  in  $M$ . Hence the sufficiency of the theorem conditions follows from Theorem 27.

□

### 3 The trace-positive quadratic polynomials

The results of the preceding section motivate the study of trace-positive self-adjoint quadratic polynomials  $f = \alpha e + \sum_{i \neq j} \alpha_{ij} u_i^* u_j$  in unitary generators  $u_1, \dots, u_n$ , i.e. polynomials having the property that  $\text{Tr}(f(V_1, \dots, V_n)) \geq 0$  for every  $m \geq 1$  and every  $n$ -tuple of unitary matrices  $V_1, \dots, V_n \in U(m)$ . If  $A$  denotes the matrix

$$\begin{pmatrix} \alpha/n & \alpha_{12} & \dots & \alpha_{1n} \\ \overline{\alpha_{12}} & \alpha/n & \dots & \alpha_{2n} \\ \dots & \dots & \dots & \dots \\ \overline{\alpha_{1n}} & \overline{\alpha_{2n}} & \dots & \alpha/n \end{pmatrix}$$

then  $\text{Tr } f(U_1, \dots, U_n) \geq 0$  can be expressed as positivity of the sum of all entries of the Schur product  $A \circ X$  where  $X = [\text{tr}(U_i^* U_j)]_{ij}$ .

Thus the trace-positive polynomials  $f$  can be characterized as those for which the sum of all entries of  $A \circ X$  for all  $X \in K_n := \{[\text{tr}(U_i^* U_j)]_{ij} \mid m \geq 1, U_1, \dots, U_n \in$

$U(m)\}$ . Thus our primary objective is to describe the sets  $K_n \subseteq M_n(\mathbb{C})$ . Note that in the case  $A$  is positive semidefinite we have  $f \in \Sigma^2(\mathcal{F})$ . Indeed in this case  $A$  is a sum of rank one positive semidefinite matrices  $A = \sum_s (\beta_{s,1}, \dots, \beta_{s,n})^T (\beta_{s,1}, \dots, \beta_{s,n})$  and hence  $f = \sum_s (\sum_j \beta_{s,j} u_j)^* (\sum_j \beta_{s,j} u_j)$ . We will also be interested in real analog of the sets  $K_n$ , i.e. the sets  $K_n(\mathbb{R}) = K_n \cap M_n(\mathbb{R})$ . Note that the sets of the traces of monomials of unitary operators and their asymptotic properties in the context of Connes' embedding problem also studied in [29] and [27].

A self-adjoint matrix  $A$  such that  $f = (u_1^{-1}, \dots, u_n^{-1}) A (u_1, \dots, u_n)^T$  is defined uniquely except for the diagonal entries. This motivates the following definition. We will call  $A$  and  $B$  *diagonally equivalent* and write  $A \stackrel{d}{\sim} B$  if  $A - B$  is a diagonal matrix with vanishing trace.

**Definition 29.** Let  $S \subseteq M_n(\mathbb{C})$  and  $A \in M_n(\mathbb{C})$  be self-adjoint. We say that  $A$  is  $S$ -positive and denote  $A \geq_S 0$  if there is self-adjoint  $B$  such that  $A \stackrel{d}{\sim} B$  and

$$\sum_{ij} b_{ij} s_{ij} \geq 0$$

for all  $s \in S$ .

The three natural choices for  $S$  will be

$$F_n = \{(t_{ij}) | t_{jj} = 1 \text{ and } |t_{ij}| \leq 1 \text{ for all } i, j\},$$

$P_n \subset F_n$  consisting of positive matrices and the set  $K_n \subset F_n$ . Clearly, an self-adjoint matrix  $A = [a_{ij}]$  is  $K_n$ -positive iff  $f = \sum_i a_{ii} e + \sum_{i \neq j} a_{ij} u_i^* u_j$  is a trace positive quadratic polynomial. Note that if

$$A \geq_{F_n} 0$$

then

$$\mathrm{Tr} A \geq \sum_{i \neq j} |a_{ij}|$$

and hence  $A \stackrel{d}{\sim} B$  for some diagonally dominant matrix  $B$ . In this case polynomial  $f = (u_1^{-1}, \dots, u_n^{-1})A(u_1, \dots, u_n)^T$  is a sum of hermitian squares. However if  $A \geq_{P_n} 0$  then  $A$  need not be diagonally equivalent to positive matrix. Note that for the three choices of  $S$  mentioned above one can use equality instead of diagonal equivalence since diagonal entries of elements in  $S$  equal to 1.

The following lemma gives a description of cyclically equivalent quadratic polynomials.

**Lemma 30.** For every matrix  $A$  the element  $(u_1^{-1}, \dots, u_n^{-1})A(u_1, \dots, u_n)^T$  is cyclically equivalent to

$$\sum_k g_k^{-1}(u_1^{-1}, \dots, u_n^{-1})A_{g_k}(u_1, \dots, u_n)^T g_k \quad (4.4)$$

for any finite collection  $g_1, \dots, g_k \in \mathbb{F}_\infty$  and any matrices  $A_g$  such that

$$\sum_k A_{g_k} \stackrel{d}{\sim} A. \quad (4.5)$$

Any element  $g \in \mathcal{F}$  such that  $g \stackrel{cyc}{\sim} f$  is of the form (4.4) for some matrices satisfying (4.5). Moreover for self-adjoint  $g$  matrices  $A_g$  can also be chosen to be self-adjoint.

*Proof.* The lemma follows from the following easy observation. For any  $w_1$  and  $w_2$  in  $\mathbb{F}_\infty$  the element  $w_1 - w_2$  is a commutator  $ab - ba$  for some  $a, b \in \mathbb{F}_\infty$  if and only if  $w_1$  and  $w_2$  are conjugated. Hence  $K$  consists of finitely supported sums of the form

$$\sum_j \sum_k \alpha_{jk} g_k^{-1} w_j g_k$$

where  $w_j, g_k$  belong to  $\mathbb{F}_\infty$  and  $\sum_k \alpha_{jk} = 0$  for all  $j$ .  $\square$

#### 4 The Clifford algebras and positive polynomials with real coefficients

For a real Hilbert space  $V$  there is a unique associative algebra  $\mathcal{C}(V)$  with a linear embedding  $J : V \rightarrow \mathcal{C}(V)$  with generating range and such that for all  $x, y \in V$

$$J(x)J(y) + J(y)J(x) = 2 \langle x, y \rangle. \quad (4.6)$$

The algebra  $\mathcal{C}(V)$  is called Clifford algebras associated to  $V$ . Clifford algebra can be realized on a Hilbert space such that for every  $x \in V$  with  $\|x\| = 1$  operator  $J(x)$  is symmetry, i.e.  $J(x)^* = J(x)$  and  $J(x)^2 = I$ . To see this consider Pauli matrices

$$U = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, Q = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Clearly  $U$  and  $Q$  are self-adjoint unitary matrices and  $U^2 = I, Q^2 = I, QU + UQ = 0$ . Then matrices  $Q_j = U \otimes \dots \otimes U \otimes Q \otimes I \otimes I \dots$  are symmetries and  $\{Q_i, Q_j\} = 2\delta_{ij}I$ . Hence operator  $J(x) = \sum_j x_j Q_j$  is also a symmetry for unit real vector  $x$ . For further properties of Clifford algebras we refer to the books [6] and [25].

**Theorem 31.** For every real correlation matrix  $P \in M_n(\mathbb{R})$  there is  $n$ -tuple of symmetries  $S_1, \dots, S_n$  in finite dimensional real Hilbert space s.t.  $P = [\text{tr}(S_i^* S_j)]_{ij}$ .

*Proof.* Every correlation  $n \times n$ -matrix  $P$  is a Gram matrix for a system of unit vectors  $x_1, \dots, x_n$ , i.e.  $P = [\langle x_i, x_j \rangle]_{ij}$ . Taking Clifford symmetries  $S_j = J(x_j)$  as in the paragraph preceding the theorem we see that  $P = [\text{tr}(S_i^* S_j)]_{ij}$ .  $\square$

**Proposition 32.** For every  $n \geq 1$  the closure  $T_n(\mathbb{R})$  of the set of matrices

$$\{[\tau(U_i^* U_j)]_{ij} | U_1, \dots, U_n \in \mathcal{U}(M)\}$$

does not depend on real type  $\text{II}_1$  von Neumann algebra  $(M, \tau)$ .

If self-adjoint  $f(u_1, \dots, u_n) \in \mathcal{F}$  has real coefficients and possess property that for every  $n$ -tuple of unitary matrices  $U_1, \dots, U_n \in U(m)$  we have  $\text{tr}(f(U_1, \dots, U_n)) \geq 0$  then for every  $\varepsilon > 0$ ,  $\varepsilon e + f \stackrel{cyc}{\approx} g$  for some  $g \in \left\{ \sum_{j=1}^m g_j^* g_j \mid m \in \mathbb{N}, g_j \in \mathbb{R}[\mathbb{F}_\infty] \right\}$ .

*Proof.* Since every  $\text{II}_1$  factor contains matrix algebras of arbitrary size we see that  $T_n(\mathbb{R})$  coincides with the set of correlation matrices. The last statement follows from Lemma 23.  $\square$

**Corollary 33.** If quadratic  $f \in \mathcal{F}$ ,  $f(u_1, \dots, u_n) = \alpha + \sum_{i \neq j} \alpha_{ij} u_i^* u_j$  is such that

$$\text{Tr}(f(U_1, \dots, U_n)) = 0$$

for all unitary matrices  $U_1, \dots, U_n$  then  $f = 0$ .

*Proof.* For every  $k \neq j$  and  $t \in [0, 1]$  the matrix  $P_1 = I + (E_{kj} + E_{jk})t$  is a real correlation matrix. Hence by the theorem there are unitary matrices  $U_1, \dots, U_n$  such that  $P_1 = [\text{tr}(U_t^* U_s)]_{ts}$ . Then the matrix  $P_2 = I + (iE_{kj} - iE_{jk})t$  is equal to  $[\text{tr}(V_t^* V_s)]_{ts}$  where  $V_t = U_t$  for  $t \neq j$  and  $V_j = iU_j$  are unitary matrices. Hence  $\alpha + (\alpha_{kj} + \alpha_{jk})t = 0$  and  $\alpha + (\alpha_{kj} - \alpha_{jk})it = 0$ . From which follows that  $\alpha = \alpha_{kj} = 0$  and hence  $f = 0$ .  $\square$

## CHAPTER V

### CONCLUSIONS

In this dissertation we study different questions and approaches in operator algebra theory. Below we outline results and state some open problems.

The first part of the dissertation devoted to study logmodular subalgebra of  $M_n(\mathbb{C})$ . The decomposition of Cholesky shows that an algebra of block upper triangular matrices is logmodular in  $M_n(\mathbb{C})$ . It was proved by V. Paulsen and M. Raghupathi in [23] that if  $\mathcal{A} \subseteq M_n(\mathbb{C})$  is logmodular and contains the diagonal matrices  $D_n$  then it is unitary equivalent to an algebra of block upper triangular matrices. It was conjectured that algebras of block upper triangular matrices are the only subalgebras on  $M_n(\mathbb{C})$  that have the logmodularity property. In we prove this in the affirmative:

**Theorem 34.** If  $\mathcal{A} \subseteq M_n(\mathbb{C})$  is logmodular then  $\mathcal{A}$  is unitary equivalent to an algebra of block upper triangular matrices.

It was proved in [22] that all contractive homomorphisms of an algebra of block upper triangular matrices are completely contractive. Thus there are no logmodular subalgebras in  $M_n(\mathbb{C})$  that can provide us with examples of contractive but not completely contractive homomorphisms.

As a possible example of a logmodular subalgebra with contractive but not-completely contractive homomorphism could be  $H^\infty(\mathbb{D})$ , considered as logmodular subalgebra of  $L^\infty(\mathbb{T})$ . In comparison, the case of the disk algebra  $\mathbb{A}(\mathbb{D}) \subseteq H^\infty(\mathbb{D})$  is well understood. Namely, every contractive homomorphism of  $\mathbb{A}(\mathbb{D})$  is extendable to a contractive homomorphism of  $C(\mathbb{T})$  and thus it is completely contractive. This can be deduced from the fact that  $\mathbb{A}(\mathbb{D})$  is Dirichlet subalgebra of  $C(\mathbb{T})$ , i.e.  $\mathbb{A}(\mathbb{D}) + \overline{\mathbb{A}(\mathbb{D})}$



is uniformly dense in  $C(\mathbb{T})$ . However the closure of  $H^\infty(\mathbb{D}) + \overline{H^\infty(\mathbb{D})}$  in  $L^\infty(\mathbb{T})$  is not an algebra. Moreover the ultraproduct of  $\mathbb{A}(\mathbb{D})$  which contains  $H^\infty(\mathbb{D})$  isometrically is not Dirichlet. There are a lot of indications that contractive homomorphisms of  $H^\infty(\mathbb{D})$  are completely contractive or at least completely bounded. In [23] and [10] it was proved that all *row-* and *column-contractive* homomorphisms are completely contractive. By a result of Bourgain, [5], every contractive homomorphism is *row* and *column* bounded. However the technique of Paulsen, Raghupathi [23] and Foias, Suciuc [10] together with result of Bourgain, are not enough to derive that every contractive homomorphism is completely bounded, which leaves the question still open.

In the second part we consider an approach to Connes embedding problem. Namely, we consider the following theorem Kirchberg, which says that in order to show that a finite von Neumann algebra  $\mathcal{M}$  with faithful tracial state  $\tau$  embeds in  $R^\omega$ , it would be enough to show that for all  $n$ , all unitary elements  $U_1, \dots, U_n$  in  $\mathcal{M}$  and all  $\varepsilon > 0$ , there is  $k \in \mathbf{N}$  and there are  $k \times k$  unitary matrices  $V_1, \dots, V_n$  such that  $|\tau(U_i^* U_j) - \text{tr}_k(V_i^* V_j)| < \varepsilon$  for all  $i, j \in \{1, \dots, n\}$ , where  $\text{tr}_k$  is the normalized trace on  $M_k(\mathbf{C})$ . It is, therefore, of interest to consider the set of possible second-order mixed moments of unitaries in such  $(\mathcal{M}, \tau)$  or, equivalently, of unitaries in  $C^*$ -algebras with respect to tracial states.

Let  $\mathcal{G}_n$  be the set of all  $n \times n$  matrices  $X$  of the form

$$X = (\tau(U_i^* U_j))_{1 \leq i, j \leq n} \quad (5.1)$$

as  $(U_1, \dots, U_n)$  runs over all  $n$ -tuples of unitaries in all  $C^*$ -algebras  $A$  possessing a faithful tracial state  $\tau$ . Let  $\mathcal{F}_n$  be the closure of the set

$$\{(\text{tr}_k(V_i^* V_j))_{1 \leq i, j \leq n} \mid k \in \mathbf{N}, V_1, \dots, V_n \in \mathcal{U}_k\},$$

where  $\mathcal{U}_k$  is the group of  $k \times k$  unitary matrices.

A *correlation matrix* is a complex, positive semidefinite matrix having all diagonal entries equal to 1. Let  $\Theta_n$  be the set of all  $n \times n$  correlation matrices. Clearly, we have

$$\mathcal{F}_n \subseteq \mathcal{G}_n \subseteq \Theta_n.$$

We prove several properties of this sets:

**Proposition 10.** *For each  $n$ ,*

- (i)  $\mathcal{F}_n$  and  $\mathcal{G}_n$  are invariant under conjugation with  $n \times n$  diagonal unitary matrices and permutation matrices,
- (ii)  $\mathcal{F}_n$  and  $\mathcal{G}_n$  are compact, convex subsets of  $\Theta_n$ ,
- (iii)  $\mathcal{F}_n$  and  $\mathcal{G}_n$  are closed under taking Schur products of matrices.

Since the sets  $\mathcal{F}_n$  and  $\mathcal{G}_n$  are convex it is fruitful to describe their extreme points. Firstly we describe extreme points of  $\Theta_n$ :

**Proposition 11.** *Let  $X \in \Theta_n$ . If  $\text{rank}(X) = r$  then there exists a frame operator  $F = (f_1, \dots, f_n)$  with unit vectors  $f_j \in \mathbb{C}^r$  such that  $X = F^*F$ . Suppose  $f_1, \dots, f_n$  is a frame consisting of  $n$  unit vectors in  $\mathbb{C}^r$ , where  $r = \text{rank}(X)$ , so that  $X = F^*F$  with  $F = (f_1, \dots, f_n)$  is the corresponding frame operator. (See Lemma 10.) Then  $X$  is an extreme point of  $\Theta_n$  if and only if the only  $r \times r$  self-adjoint matrix  $Z$  satisfying  $\langle Zf_j, f_j \rangle = 0$  for all  $j \in \{1, \dots, n\}$  is the zero matrix.*

Since every  $X \in \Theta_n$  with  $\text{rank}(X) = 1$  has the form  $X = (e^{i(\phi_i - \phi_j)})$  we have  $\text{ext}(F_3) = \text{ext}(\Theta_3)$  and thus  $F_3 = \Theta_3$ .

Let  $X = F^*F$  with frame operator given by

$$f_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad f_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad f_3 = \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}, \quad f_4 = \begin{pmatrix} i/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}$$

Then  $X \in \text{ext}(\Theta_4)$  and  $\text{rank}(X) = 2$ , thus  $X \notin \text{ext}(F_4)$ . Therefore we have  $\Theta_n \neq F_n$  for  $n \geq 4$ .

The description of sets  $\mathcal{F}_n \subseteq \mathcal{G}_n \subseteq \Theta_n$  with real coefficients is completely different. The proof of the following theorem is based on representations of the Clifford algebra.

**Theorem 35.** For every  $n \in \mathbf{N}$ , we have

$$M_n(\mathbb{R}) \cap \Theta_n \subseteq \mathcal{F}_n.$$

In the last chapter we consider recent result of I. Klep and M. Schweighofer who established that Connes' embedding problem has the following equivalent algebraic reformulation.

Let  $f(X_1, \dots, X_m)$  be a self-adjoint element in a free associative algebra  $\mathbb{K}\langle \overline{X} \rangle$  with countable family of self-adjoint generators  $\overline{X} = \{X_1, X_2, \dots\}$ , where  $\mathbb{K} = \mathbb{R}$  or  $K = \mathbb{C}$ . If  $\text{tr}(f(A_1, \dots, A_m)) \geq 0$  for any  $n$  and family of self-adjoint contractive matrices  $A_1, \dots, A_m \in M_n(\mathbb{K})$  then  $f$  has the property that for every  $\varepsilon > 0$  we have  $\varepsilon e + f = g + c$  where  $c$  is a sum of commutators in  $\mathbb{K}\langle \overline{X} \rangle$ ,  $g$  belongs to quadratic module generated by  $1 - X_i^2$  and  $e$  is the unit in  $\mathbb{K}\langle \overline{X} \rangle$ . Recall that a *quadratic module* is the smallest subset of  $\mathbb{K}\langle \overline{X} \rangle$  containing unit, closed under addition and conjugation  $x \rightarrow g^* x g$  by arbitrary  $g \in \mathbb{K}\langle \overline{X} \rangle$ .

We consider the group  $*$ -algebra  $\mathcal{F}$  of the countably generated free group  $\mathbb{F}_\infty = \langle u_1, u_2, \dots \rangle$  instead of  $\mathbb{K}\langle \overline{X} \rangle$ . One reason is that we can use a more standard and well known set of hermitian squares  $\{g^* g | g \in \mathcal{F}\}$  instead of quadratic module  $M$  and the second that we can bound the degree of polynomials  $f$  in the above reformulation by 2. This modification provides the following.

**Theorem.** Connes' embedding conjecture is true iff for any self-adjoint  $f \in \mathcal{F}$  of the

form  $f(u_1, \dots, u_n) = \alpha e + \sum_{i \neq j} \alpha_{ij} u_i^* u_j$  condition

$$\text{Tr}(f(V_1, \dots, V_n)) \geq 0 \quad (5.2)$$

for every  $m \geq 1$  and every  $n$ -tuple of unitary matrices  $V_1, \dots, V_n \in U(m)$  implies that for every  $\varepsilon > 0$ ,  $\varepsilon e + f = g + c$  where  $c$  is a sum of commutators and  $g$  is a sum of Hermitian squares.

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The typist for this dissertation was Kateryna Iushchenko.